Tensor Satisfying Binary Law about the First, Second, Third-Order Covariant Derivative of the Contravariant Vector

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Abstract

If all coordinate systems satisfy Binary Law, I have already reported that \( A_{\mu}^\nu = \frac{\partial A_{\mu}^\nu}{\partial x^\nu} \) is established for the first-order covariant derivative of the contravariant vector. In addition, if all coordinate systems satisfy Binary Law, I have already reported that \( A_{\mu,\nu}^{\mu,\nu} = \frac{\partial^2 A_{\mu}^\nu}{\partial x^\nu \partial x^\nu} \), \( A_{\mu,\nu_{\nu}}^{\mu,\nu_{\nu}} = \frac{\partial^3 A_{\mu}^\nu}{\partial x^\nu \partial x^\nu \partial x^\nu} \) is established for the second, third-order covariant derivative of the contravariant vector. I report proof put in order based on proof in the above newly in this article. Furthermore, I report “Property of Tensor satisfying Binary Law about the third-order Covariant Derivative of the contravariant Vector” in this article.

Keywords

Tensor; Covariant Derivative of the contravariant Vector; Binary Law; Tensor Satisfying Binary Law

Introduction

I have already reported Definition 9, Definition 10, Definition 11. I remade proof of the derivation of Tensor satisfying Binary Law: \( A_{\mu}^\nu = \frac{\partial A_{\mu}^\nu}{\partial x^\nu} \) based on proof in Definition 9. Similarly, I remade proof of the derivation of Tensor satisfying Binary Law: \( A_{\mu,\nu}^{\mu,\nu} = \frac{\partial^2 A_{\mu}^\nu}{\partial x^\nu \partial x^\nu} \) based on proof in Definition10. Similarly, I remade proof of the derivation of Tensor satisfying Binary Law: \( A_{\mu,\nu_{\nu}}^{\mu,\nu_{\nu}} = \frac{\partial^3 A_{\mu}^\nu}{\partial x^\nu \partial x^\nu \partial x^\nu} \) based on proof in Definition 11. I report the proof provided newly mentioned above in this article.

2 Definition

Definition 1 \( \overline{x^\mu} \neq x^\mu, \overline{x^\nu} \neq x^\nu, \overline{x^\mu} = x^\nu, \overline{x^\nu} = x^\mu \) is established. [1] I named

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\[ x^\mu \neq x^{\nu}, x^\nu \neq x^\mu, \overline{x}^\mu = x^\nu, \overline{x}^\nu = x^\mu \] “Binary Law”. [1]

**Definition 2** If \( \overline{x}^\mu \neq x^\mu, \overline{x}^\nu \neq x^\nu, \overline{x}^\mu = x^\nu, \overline{x}^\nu = x^\mu \) is established, \( x^\nu = x^\nu \) is established. [1]

**Definition 3** If \( \overline{x}^\mu \neq x^\mu, \overline{x}^\nu \neq x^\nu, \overline{x}^\mu = x^\nu, \overline{x}^\nu = x^\mu \) is established, \( x^\nu = -x^\mu \) is established. [1]

**Definition 4** If all coordinate systems \( x^\mu, x^\nu, x^\sigma, x^\lambda, \ldots \) satisfies \( \overline{x}^\mu \neq x^\mu, \overline{x}^\nu \neq x^\nu, \overline{x}^\mu = x^\nu, \overline{x}^\nu = x^\mu \), all coordinate systems \( x^\mu, x^\nu, x^\sigma, x^\lambda, \ldots \) shifts to only two of \( x^\mu, x^\nu \). [1]

**Definition 5** \( g^\mu_\mu = 1, g^\nu_\nu = 0 (\mu \neq \nu) \) is establishment. [3]

**Definition 6** The first-order covariant derivative of the contravariant vector satisfies

\[
A^\mu_\nu = \frac{\partial A^\mu}{\partial x^\nu} + A^\mu_\tau \Gamma^\tau_\nu = \frac{\partial A^\mu}{\partial x^\nu} + A^\tau_\nu \Gamma^\tau_\mu = \frac{1}{2} g^{\mu_\mu} \left( \frac{\partial g^\nu_\upsilon}{\partial x^\xi} + \frac{\partial g^\upsilon_\xi}{\partial x^\nu} - \frac{\partial g^\xi_\upsilon}{\partial x^\nu} \right). [4]
\]

**Definition 7** The second-order covariant derivative of the contravariant vector satisfies

\[
A^\mu_\nu = \frac{\partial A^\mu_\nu}{\partial x^\sigma} + A^\mu_\nu \Gamma^\sigma_\nu = \frac{\partial A^\mu_\nu}{\partial x^\sigma} + A^\tau_\nu \Gamma^\tau_\mu = \frac{1}{2} g^{\mu_\mu} \left( \frac{\partial g^\nu_\upsilon}{\partial x^\xi} + \frac{\partial g^\upsilon_\xi}{\partial x^\nu} - \frac{\partial g^\xi_\upsilon}{\partial x^\nu} \right) + \frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} g^{\mu_\mu} \left( \frac{\partial g^\nu_\upsilon}{\partial x^\xi} + \frac{\partial g^\upsilon_\xi}{\partial x^\nu} - \frac{\partial g^\xi_\upsilon}{\partial x^\nu} \right) + \frac{1}{2} g^{\mu_\mu} \left( \frac{\partial g^\nu_\upsilon}{\partial x^\xi} + \frac{\partial g^\upsilon_\xi}{\partial x^\nu} - \frac{\partial g^\xi_\upsilon}{\partial x^\nu} \right) \frac{1}{2} g^{\nu_\nu} \left( \frac{\partial g^{\mu_\sigma}}{\partial x^\tau} + \frac{\partial g^{\sigma_\tau}}{\partial x^\mu} - \frac{\partial g^{\tau_\sigma}}{\partial x^\nu} \right) [4]
\]

**Definition 8** The third-order covariant derivative of the contravariant vector satisfies

\[
A^\mu_\nu = \frac{\partial A^\mu_\nu}{\partial x^\sigma} + A^\mu_\nu \Gamma^\sigma_\nu = \frac{\partial A^\mu_\nu}{\partial x^\sigma} + A^\tau_\nu \Gamma^\tau_\mu = \frac{1}{2} g^{\mu_\mu} \left( \frac{\partial g^\nu_\upsilon}{\partial x^\xi} + \frac{\partial g^\upsilon_\xi}{\partial x^\nu} - \frac{\partial g^\xi_\upsilon}{\partial x^\nu} \right) + \frac{1}{2} g^{\mu_\mu} \left( \frac{\partial g^\nu_\upsilon}{\partial x^\xi} + \frac{\partial g^\upsilon_\xi}{\partial x^\nu} - \frac{\partial g^\xi_\upsilon}{\partial x^\nu} \right) + \frac{1}{2} g^{\mu_\mu} \left( \frac{\partial g^\nu_\upsilon}{\partial x^\xi} + \frac{\partial g^\upsilon_\xi}{\partial x^\nu} - \frac{\partial g^\xi_\upsilon}{\partial x^\nu} \right) + \frac{1}{2} g^{\mu_\mu} \left( \frac{\partial g^\nu_\upsilon}{\partial x^\xi} + \frac{\partial g^\upsilon_\xi}{\partial x^\nu} - \frac{\partial g^\xi_\upsilon}{\partial x^\nu} \right) [4]
\]
\[
\begin{align*}
\frac{\partial}{\partial x^\lambda} \left( \frac{\partial A^\mu}{\partial x^\rho} + A^\mu \frac{\partial}{\partial x^\rho} \right) + \left( \frac{\partial A^\mu}{\partial x^\rho} + A^\mu \frac{\partial}{\partial x^\rho} \right) \Gamma^\lambda_{\mu \rho} - \left( \frac{\partial A^\mu}{\partial x^\rho} + A^\mu \frac{\partial}{\partial x^\rho} \right) \Gamma^\lambda_{\mu \rho} \\
+ \frac{\partial}{\partial x^\sigma} \left( \frac{\partial A^\mu}{\partial x^\nu} + A^\mu \frac{\partial}{\partial x^\nu} \right) + \frac{\partial A^\mu}{\partial x^\nu} + A^\mu \frac{\partial}{\partial x^\nu} \right) \Gamma^\lambda_{\mu \nu} - \frac{\partial A^\mu}{\partial x^\nu} + A^\mu \frac{\partial}{\partial x^\nu} \right) \Gamma^\lambda_{\mu \nu} \\
- \frac{\partial}{\partial x^\sigma} \left( \frac{\partial A^\mu}{\partial x^\nu} + A^\mu \frac{\partial}{\partial x^\nu} \right) + \frac{\partial A^\mu}{\partial x^\nu} + A^\mu \frac{\partial}{\partial x^\nu} \right) \Gamma^\lambda_{\mu \nu} - \frac{\partial A^\mu}{\partial x^\nu} + A^\mu \frac{\partial}{\partial x^\nu} \right) \Gamma^\lambda_{\mu \nu} \\
- \frac{\partial}{\partial x^\lambda} \left( \frac{\partial A^\mu}{\partial x^\nu} + A^\mu \frac{\partial}{\partial x^\nu} \right) + \frac{\partial A^\mu}{\partial x^\nu} + A^\mu \frac{\partial}{\partial x^\nu} \right) \Gamma^\lambda_{\mu \nu} - \frac{\partial A^\mu}{\partial x^\nu} + A^\mu \frac{\partial}{\partial x^\nu} \right) \Gamma^\lambda_{\mu \nu} \\
= \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\sigma \partial x^\lambda} g^{\rho \lambda} + \frac{\partial}{\partial x^\sigma} \left( \frac{\partial A^\mu}{\partial x^\rho} + A^\mu \frac{\partial}{\partial x^\rho} \right) g^{\sigma \lambda} - \frac{\partial A^\mu}{\partial x^\rho} + A^\mu \frac{\partial}{\partial x^\rho} \right) g^{\sigma \lambda} \\
+ \frac{\partial}{\partial x^\rho} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
+ \frac{\partial^2 A^\mu}{\partial x^\rho \partial x^\sigma} g^{\rho \lambda} + \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
+ \frac{\partial}{\partial x^\rho} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
+ \frac{A^\mu}{2} g^{\rho \lambda} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
- \frac{A^\mu}{2} g^{\rho \lambda} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
- \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{A^\mu}{2} g^{\rho \lambda} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
- \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{A^\mu}{2} g^{\rho \lambda} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
= \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\sigma \partial x^\lambda} g^{\rho \lambda} + \frac{\partial}{\partial x^\sigma} \left( \frac{\partial A^\mu}{\partial x^\rho} + A^\mu \frac{\partial}{\partial x^\rho} \right) g^{\sigma \lambda} - \frac{\partial A^\mu}{\partial x^\rho} + A^\mu \frac{\partial}{\partial x^\rho} \right) g^{\sigma \lambda} \\
+ \frac{\partial}{\partial x^\rho} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
+ \frac{\partial^2 A^\mu}{\partial x^\rho \partial x^\sigma} g^{\rho \lambda} + \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
+ \frac{\partial}{\partial x^\rho} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
+ \frac{A^\mu}{2} g^{\rho \lambda} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
- \frac{A^\mu}{2} g^{\rho \lambda} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
- \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{A^\mu}{2} g^{\rho \lambda} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
- \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{A^\mu}{2} g^{\rho \lambda} \left( \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) - \frac{\partial A^\mu}{\partial x^\sigma} + A^\mu \frac{\partial}{\partial x^\sigma} \right) g^{\rho \lambda} \\
\end{align*}
\]
\(-A^{\mu} \frac{1}{2} g^{\epsilon \eta} \left( \frac{\partial g_{\epsilon \mu}}{\partial x^\eta} - \frac{\partial g_{\eta \mu}}{\partial x^\eta} \right) + \frac{\partial A^{\mu}}{\partial x^\eta} \frac{1}{2} g^{\epsilon \eta} \left( \frac{\partial g_{\epsilon \mu}}{\partial x^\eta} + \frac{\partial g_{\eta \mu}}{\partial x^\eta} - \frac{\partial g_{\epsilon \eta}}{\partial x^\eta} \right) + A^{\nu} \frac{1}{2} g^{\epsilon \eta} \left( \frac{\partial g_{\epsilon \nu}}{\partial x^\eta} - \frac{\partial g_{\eta \nu}}{\partial x^\eta} \right) + A^{\sigma} \frac{1}{2} g^{\epsilon \eta} \left( \frac{\partial g_{\epsilon \sigma}}{\partial x^\eta} + \frac{\partial g_{\eta \sigma}}{\partial x^\eta} - \frac{\partial g_{\epsilon \eta}}{\partial x^\eta} \right) \). [4]

**Definition 9** If all coordinate systems \(x^{\mu'}, x^{\nu'}, x^{\sigma'}, \ldots\) satisfies \(x^{\mu'} \neq x^{\mu}, x^{\nu'} \neq x^{\nu}, x^{\sigma'} = x^{\sigma}, \overline{x}^{\mu'} = x^{\mu}, \overline{x}^{\nu'} = x^{\nu}, \overline{x}^{\sigma'} = x^{\sigma}\), covariant differentiation for Contravariant Vector \(A^{\mu'}\) behave like a covariant differentiation for Scalar \(S_{\nu'}\). [1]

**Definition 10** If \(x^{\mu} \neq x^{\nu}, x^{\sigma}, \ldots\) is established, \(A^{\mu'} = \frac{\partial^2 A^{\mu}}{\partial x^{\nu} \partial x^{\eta}}\) is established. [2]

**Definition 11** If \(x^{\mu} \neq x^{\nu}, x^{\sigma}, \ldots\) is established, \(A^{\mu'}_{\nu \sigma} = \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\tau}}\) is established. [2]

**Definition 12** I express \(x^{\mu}\) satisfying \(\frac{\partial^3 x^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\tau}} = M \ln \left\{ x^{\mu} \right\}_M\).

3 Tensor satisfying Binary Law about the first-order Covariant Derivative of the Contravariant Vector

**Proposition 1** Definition6 isn’t an equation of the tensor satisfying Binary Law.

Proof: If all coordinate systems satisfies Binary Law, I get

\[ A^{\mu'} = \frac{\partial A^{\mu}}{\partial x^{\nu}} + A^{\nu} \frac{1}{2} g^{\epsilon \eta} \left( \frac{\partial g_{\epsilon \nu}}{\partial x^{\eta}} + \frac{\partial g_{\eta \nu}}{\partial x^{\eta}} - \frac{\partial g_{\epsilon \eta}}{\partial x^{\eta}} \right) + A^{\sigma} \frac{1}{2} g^{\epsilon \eta} \left( \frac{\partial g_{\epsilon \sigma}}{\partial x^{\eta}} + \frac{\partial g_{\eta \sigma}}{\partial x^{\eta}} - \frac{\partial g_{\epsilon \eta}}{\partial x^{\eta}} \right) \] (1)

From Definition 6. I rewrite dummy index of (1) in the index which isn’t used in this term and must get

\[ A^{\mu'} = \frac{\partial A^{\mu}}{\partial x^{\nu}} + A^{\alpha} \frac{1}{2} g^{\epsilon \eta} \left( \frac{\partial g_{\epsilon \alpha}}{\partial x^{\eta}} + \frac{\partial g_{\eta \alpha}}{\partial x^{\eta}} - \frac{\partial g_{\epsilon \eta}}{\partial x^{\eta}} \right) \] (2)
\[ A^\mu_\nu = \frac{\partial A^\mu}{\partial x_\nu} + A_\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x_\nu} \right) \]  

(3)

because (1) must be an equation of the tensor. I get the conclusion that (2), (3) doesn’t satisfy Binary law from Definition 4. If (2), (3) satisfies Binary Law, Definition 6 is an equation of the tensor satisfying Binary Law here. However, I get the conclusion that Definition 6 isn’t an equation of the tensor satisfying Binary Law because (2), (3) doesn’t satisfy Binary Law.

- End Proof -

**Proposition 2** \( A^\mu_\nu = \frac{\partial A^\mu}{\partial x^\nu} \) is the equation that was rewritten so that Definition 6 is an equation of the tensor satisfying Binary Law.

Proof: I rewrite index \( \nu \) of the coordinate systems of (2), (3) by Definition 3 and get

\[ -A^\mu_\nu = -\frac{\partial A^\mu}{\partial x^\mu} - A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\nu} \right), \]

(4)

\[ -A^\mu_\nu = -\frac{\partial A^\mu}{\partial x^\mu} - A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\nu} \right). \]

(5)

There isn’t index \( \nu \) in the second term of the right side of (4), (5) at all here. Therefore, I rewrite dummy index \( \sigma \) of (4), (5) in dummy index \( \nu \) and can get

\[ -A^\mu_\nu = -\frac{\partial A^\mu}{\partial x^\mu} - A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\nu} \right), \]

(6)

\[ -A^\mu_\nu = -\frac{\partial A^\mu}{\partial x^\mu} - A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\nu} \right). \]

(7)

And I get

\[ -A^\mu_\nu = -\frac{\partial A^\mu}{\partial x^\mu} \]

(8)

from (6), (7) in consideration of Definition 5. In addition,

\[ A^\mu_\nu = \frac{\partial A^\mu}{\partial x^\nu} \]

(9)

can rewrite (8) using Definition 2 because the second term doesn’t exist in the right side of (8). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (2), (3) by Definition 2 and get

\[ A^{\mu\nu} = \frac{\partial A^\mu}{\partial x_\mu} + A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x_\nu} \right), \]

(10)

\[ A^{\mu\nu} = \frac{\partial A^\mu}{\partial x_\mu} + A^\sigma \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x_\nu} \right). \]

(11)

Because two same covariant index existed in \( A^{\mu\nu} \) of (10), (11), I decided not to handle (10), (11). Because (9) doesn’t
change the form of the equation even if all coordinate systems satisfies Binary Law, I get the conclusion that (9) satisfies Binary Law. In other words, (9) is an equation of the tensor satisfying Binary Law. Because (9) is an equation provided from Definition 6, (9) is the equation that was rewritten so that Definition 6 is an equation of the tensor satisfying Binary Law.

– End Proof –

4 Tensor satisfying Binary Law about the second-order Covariant Derivative of the Contravariant Vector

Proposition 3 Definition 7 isn’t an equation of the tensor satisfying Binary Law.

Proof: If all coordinate systems satisfies Binary Law, I get

\[
A^\mu_{\nu,\sigma} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\sigma} + \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} g^{\mu \nu} \left( \frac{\partial g_{\nu,\nu}}{\partial x^\nu} + \frac{\partial g_{\nu,\nu}}{\partial x^\nu} - \frac{\partial g_{\nu,\nu}}{\partial x^\nu} \right) \right) + \frac{\partial A^\nu}{\partial x^\nu} \left( A^\nu \frac{1}{2} g^{\mu \nu} \left( \frac{\partial g_{\nu,\nu}}{\partial x^\nu} + \frac{\partial g_{\nu,\nu}}{\partial x^\nu} - \frac{\partial g_{\nu,\nu}}{\partial x^\nu} \right) \right)
\]

\[+ A^\nu \frac{1}{2} g^{\mu \nu} \left( \frac{\partial g_{\nu,\nu}}{\partial x^\nu} + \frac{\partial g_{\nu,\nu}}{\partial x^\nu} - \frac{\partial g_{\nu,\nu}}{\partial x^\nu} \right) \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\nu,\rho}}{\partial x^\rho} + \frac{\partial g_{\nu,\rho}}{\partial x^\rho} - \frac{\partial g_{\nu,\rho}}{\partial x^\rho} \right) \]

\[= \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\sigma} + \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} g^{\mu \nu} \left( \frac{\partial g_{\nu,\nu}}{\partial x^\nu} + \frac{\partial g_{\nu,\nu}}{\partial x^\nu} - \frac{\partial g_{\nu,\nu}}{\partial x^\nu} \right) \right) \]

\[+ \frac{\partial A^\nu}{\partial x^\nu} \left( A^\nu \frac{1}{2} g^{\mu \nu} \left( \frac{\partial g_{\nu,\nu}}{\partial x^\nu} + \frac{\partial g_{\nu,\nu}}{\partial x^\nu} - \frac{\partial g_{\nu,\nu}}{\partial x^\nu} \right) \right) \]

\[+ \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} g^{\mu \nu} \left( \frac{\partial g_{\nu,\nu}}{\partial x^\nu} + \frac{\partial g_{\nu,\nu}}{\partial x^\nu} - \frac{\partial g_{\nu,\nu}}{\partial x^\nu} \right) \right) \]

from Definition 7. I rewrite dummy index of (13) in the index which isn’t used in this term and must get

\[
A^\mu_{\nu,\sigma} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\sigma} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \frac{\partial g_{\nu,\nu}}{\partial x^\nu} \right) \]

\[A^\mu_{\nu,\sigma} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\sigma} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \frac{\partial g_{\nu,\nu}}{\partial x^\nu} \right) \]

\[A^\mu_{\nu,\sigma} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\sigma} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \frac{\partial g_{\nu,\nu}}{\partial x^\nu} \right) \]

because (12) must be an equation of the tensor. I get the conclusion that (16), (17), (18) doesn’t satisfy Binary Law from
Definision 4. Similarly, I rewrite dummy index of (14) in the index which isn’t used in this term and must get

\[ A_{\nu,\mu}^\mu = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{\partial g^\mu_\nu}{\partial x^\mu} \right), \quad (19) \]

\[ A_{\nu,\mu}^\mu = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{\partial g^\mu_\nu}{\partial x^\sigma} \right), \quad (20) \]

\[ A_{\nu,\mu}^\mu = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{\partial g^\mu_\nu}{\partial x^\sigma} \right), \quad (21) \]

because (12) must be an equation of the tensor. I get the conclusion that (19), (20), (21) doesn’t satisfy Binary Law from Definision 4. Similarly, I rewrite dummy index of (15) in the index which isn’t used in this term and must get

\[ A_{\nu,\mu}^\mu = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\mu}{\partial x^\nu} \left( \frac{\partial g^\mu_\nu}{\partial x^\mu} \right), \quad (22) \]

\[ A_{\nu,\mu}^\mu = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \left( \frac{\partial g^\mu_\nu}{\partial x^\sigma} \right), \quad (23) \]

\[ A_{\nu,\mu}^\mu = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \left( \frac{\partial g^\mu_\nu}{\partial x^\sigma} \right), \quad (24) \]

because (12) must be an equation of the tensor. I get the conclusion that (22), (23), (24) doesn’t satisfy Binary Law from Definision 4. If (16) ~ (24) satisfies Binary Law, Definision 7 is an equation of the tensor satisfying Binary Law here. However, I get the conclusion that Definision 7 isn’t an equation of the tensor satisfying Binary Law because (16) ~ (24) doesn’t satisfy Binary Law.

–End Proof–

Proposition 4 \( A_{\nu,\mu}^{\mu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} \) is the equation that was rewritten so that Definision 7 is an equation of the tensor satisfying Binary Law.

Proof: I rewrite index \( \nu \) of the coordinate systems of (16), (17), (18) by Definision 3 and get

\[ A_{\mu,\nu}^{\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\nu} + \frac{\partial A^\mu}{\partial x^\mu} \left( \frac{\partial g^\mu_\nu}{\partial x^\mu} \right), \quad (25) \]

\[ A_{\mu,\nu}^{\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\nu} + \frac{\partial A^\mu}{\partial x^\nu} \left( \frac{\partial g^\mu_\nu}{\partial x^\mu} \right), \quad (26) \]

\[ A_{\mu,\nu}^{\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\nu} + \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\mu_\nu}{\partial x^\sigma} \right), \quad (27) \]

There isn’t index \( \nu \) in the second term of the right side of (25), (26), (27) at all here. Therefore, I rewrite dummy index \( \sigma \) of (25), (26), (27) in dummy index \( \nu \) and can get

\[ A_{\mu,\nu}^{\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\nu} + \frac{\partial A^\mu}{\partial x^\nu} \left( \frac{\partial g^\mu_\nu}{\partial x^\nu} \right), \quad (28) \]

\[ A_{\mu,\nu}^{\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\nu} + \frac{\partial A^\mu}{\partial x^\nu} \left( \frac{\partial g^\mu_\nu}{\partial x^\nu} \right), \quad (29) \]
\[ A''_{\mu;\nu} = \frac{\partial^2 A''_{\mu}}{\partial x^\nu \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A''^{\sigma} \frac{1}{2} \frac{\partial g''_{\mu \sigma}}{\partial x^\mu} \right) \] ...

(30)

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (16), (17), (18) by Definition 2 and get

\[ A''_{\mu;\nu} = \frac{\partial^2 A''_{\mu}}{\partial x^\nu \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A''^{\sigma} \frac{1}{2} \frac{\partial g''_{\mu \sigma}}{\partial x^\sigma} \right) \] ...

(31)

\[ A''''_{\mu;\nu} = \frac{\partial^2 A''''_{\mu}}{\partial x^\nu \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A''''^{\sigma} \frac{1}{2} \frac{\partial g''''_{\mu \sigma}}{\partial x^\sigma} \right) \] ...

(32)

\[ A''''''_{\mu;\nu} = \frac{\partial^2 A''''''_{\mu}}{\partial x^\nu \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A''''''' \frac{1}{2} \frac{\partial g'''''''_{\mu \sigma}}{\partial x^\sigma} \right) \] ...

(33)

Because two same contravariant index existed in \( A''_{\mu;\nu} \) of (31), (32), (33), I decided not to handle (31), (32), (33). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (16), (17), (18) by Definition 3, Definition 2 and get

\[ -A''_{\mu} = \frac{\partial^2 A''_{\mu}}{\partial x^\mu \partial x^\nu} + \frac{\partial}{\partial x^\mu} \left( A''^{\sigma} \frac{1}{2} \frac{\partial g''_{\mu \sigma}}{\partial x^\sigma} \right) \] ...

(34)

\[ -A''''_{\mu} = \frac{\partial^2 A''''_{\mu}}{\partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\sigma} \left( A''''^{\mu} \frac{1}{2} \frac{\partial g''''_{\mu \mu}}{\partial x^\sigma} \right) \] ...

(35)

\[ -A''''''_{\mu} = \frac{\partial^2 A''''''_{\mu}}{\partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\sigma} \left( A''''''' \frac{1}{2} \frac{\partial g'''''''_{\mu \sigma}}{\partial x^\sigma} \right) \] ...

(36)

There isn’t index \( \nu \) in the second term of the right side of (34), (35), (36) at all here. Therefore, I rewrite dummy index \( \sigma \) of (34), (35), (36) in dummy index \( \nu \) and can get

\[ -A''_{\mu} = \frac{\partial^2 A''_{\mu}}{\partial x^\mu \partial x^\nu} + \frac{\partial}{\partial x^\mu} \left( A''^{\nu} \frac{1}{2} \frac{\partial g''_{\mu \nu}}{\partial x^\nu} \right) \] ...

(37)

\[ -A''''_{\mu} = \frac{\partial^2 A''''_{\mu}}{\partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\sigma} \left( A''''^{\mu} \frac{1}{2} \frac{\partial g''''_{\mu \mu}}{\partial x^\sigma} \right) \] ...

(38)

\[ -A''''''_{\mu} = \frac{\partial^2 A''''''_{\mu}}{\partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\sigma} \left( A''''''' \frac{1}{2} \frac{\partial g'''''''_{\mu \sigma}}{\partial x^\sigma} \right) \] ...

(39)

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (16), (17), (18) by Definition 2, Definition 3 and get

\[ -A''_{\mu;\nu} = \frac{\partial^2 A''_{\mu}}{\partial x^\nu \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A''^{\sigma} \frac{1}{2} \frac{\partial g''_{\mu \sigma}}{\partial x^\sigma} \right) \] ...

(40)

\[ -A''''_{\mu;\nu} = \frac{\partial^2 A''''_{\mu}}{\partial x^\nu \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A''''^{\mu} \frac{1}{2} \frac{\partial g''''_{\mu \mu}}{\partial x^\sigma} \right) \] ...

(41)

\[ -A''''''_{\mu;\nu} = \frac{\partial^2 A''''''_{\mu}}{\partial x^\nu \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A''''''' \frac{1}{2} \frac{\partial g'''''''_{\mu \sigma}}{\partial x^\sigma} \right) \] ...

(42)

There isn’t index \( \nu \) in the second term of the right side of (40), (41), (42) at all here. Therefore, I rewrite dummy index \( \sigma \) of (40), (41), (42) in dummy index \( \nu \) and can get
\[-(A_{\mu,\nu})_{,\mu} = -\frac{\partial^2 A_{\mu}}{\partial x_{\nu} \partial x_{\mu}} + \frac{\partial}{\partial x_{\mu}} \left( A_{\nu} \frac{1}{2} \left( \frac{\partial g_{\mu \nu}}{\partial x_{\mu}} \right) \right), \quad (43)\]
\[-(A_{\mu,\nu})_{,\nu} = -\frac{\partial^2 A_{\mu}}{\partial x_{\nu} \partial x_{\mu}} - \frac{\partial}{\partial x_{\mu}} \left( A_{\nu} \frac{1}{2} \left( \frac{\partial g_{\mu \nu}}{\partial x_{\nu}} \right) \right), \quad (44)\]
\[-(A_{\mu,\nu})_{,\nu} = -\frac{\partial^2 A_{\mu}}{\partial x_{\nu} \partial x_{\mu}} + \frac{\partial}{\partial x_{\nu}} \left( A_{\nu} \frac{1}{2} \left( \frac{\partial g_{\mu \nu}}{\partial x_{\mu}} \right) \right), \quad (45)\]

By the way, by establishment of \( A_{\mu,\nu} = A_{\nu,\mu} \), (37) and (43) must be equal. (37) and (43) are equal in consideration of Definition 5. Furthermore, by establishment of \( A_{\mu,\nu} = A_{\nu,\mu} \), (38) and (44) must be equal. (38) and (44) aren’t equal in consideration of Definition 5. Thus, I decide not to handle (38), (44). Furthermore, by establishment of \( A_{\mu,\nu} = A_{\nu,\mu} \), (39) and (45) must be equal. (39) and (45) aren’t equal in consideration of Definition 5. Thus, I decide not to handle (39), (45).

Similarly. I rewrite index \( \nu \) of the coordinate systems of (19), (20), (21) by Definition 3 and get
\[A^\nu_{\mu,\nu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\sigma \nu}}{\partial x^\nu} \right), \quad (46)\]
\[A^\nu_{\mu,\nu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\sigma}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g_{\sigma \nu}}{\partial x^\nu} \right), \quad (47)\]
\[A^\nu_{\mu,\nu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\sigma}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g_{\sigma \nu}}{\partial x^\nu} \right), \quad (48)\]

There isn’t index \( \nu \) in the second term of the right side of (46), (47), (48) at all here. Therefore, I rewrite dummy index \( \sigma \) of (46), (47), (48) in dummy index \( \nu \) and can get
\[A^\nu_{\mu,\nu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\nu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g_{\nu \nu}}{\partial x^\nu} \right), \quad (49)\]
\[A^\nu_{\mu,\nu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\nu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g_{\nu \nu}}{\partial x^\nu} \right), \quad (50)\]
\[A^\nu_{\mu,\nu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\nu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g_{\nu \nu}}{\partial x^\nu} \right), \quad (51)\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (19), (20), (21) by Definition 2 and get
\[A^\nu_{\mu,\nu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\nu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right), \quad (52)\]
\[A^\nu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\sigma \mu}}{\partial x^\mu} \right), \quad (53)\]
\[A^\nu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\mu} + \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\sigma \mu}}{\partial x^\mu} \right), \quad (54)\]

Because two same contravariant index existed in \( A^\nu_{\mu,\nu} \) of (52), (53), (54), I decided not to handle (52), (53), (54). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (19), (20), (21) by Definition 3, Definition 2 and get
\[-\left( A_\mu^\nu \right)^\alpha = - \frac{\partial^2 A_\mu^\nu}{\partial x^\alpha \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\alpha^\mu \right), \quad (55) \]
\[-\left( A_\mu^\nu \right)^\rho = - \frac{\partial^2 A_\mu^\nu}{\partial x^\rho \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\rho^\mu \right), \quad (56) \]
\[-\left( A_\mu^\nu \right)^\sigma = - \frac{\partial^2 A_\mu^\nu}{\partial x^\sigma \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\sigma^\mu \right), \quad (57) \]

There isn’t index \( \nu \) in the second term of the right side of (55), (56), (57) at all here. Therefore, I rewrite dummy index \( \sigma \) of (55), (56), (57) in dummy index \( \nu \) and get
\[-\left( A_\mu^\nu \right)^\mu = - \frac{\partial^2 A_\mu^\nu}{\partial x^\mu \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\mu^\mu \right), \quad (58) \]
\[-\left( A_\mu^\nu \right)^\rho = - \frac{\partial^2 A_\mu^\nu}{\partial x^\rho \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\rho^\mu \right), \quad (59) \]
\[-\left( A_\mu^\nu \right)^\sigma = - \frac{\partial^2 A_\mu^\nu}{\partial x^\sigma \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\sigma^\mu \right), \quad (60) \]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (19), (20), (21) by Definition 2, Definition 3 and get
\[-\left( A_\mu^\nu \right)^\mu = - \frac{\partial^2 A_\mu^\nu}{\partial x^\mu \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\mu^\nu \right), \quad (61) \]
\[-\left( A_\mu^\nu \right)^\rho = - \frac{\partial^2 A_\mu^\nu}{\partial x^\rho \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\rho^\nu \right), \quad (62) \]
\[-\left( A_\mu^\nu \right)^\sigma = - \frac{\partial^2 A_\mu^\nu}{\partial x^\sigma \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\sigma^\nu \right), \quad (63) \]

There isn’t index \( \nu \) in the second term of the right side of (61), (62), (63) at all here. Therefore, I rewrite dummy index \( \sigma \) of (61), (62), (63) in dummy index \( \nu \) and can get
\[-\left( A_\mu^\nu \right)^\mu = - \frac{\partial^2 A_\mu^\nu}{\partial x^\mu \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\mu^\nu \right), \quad (64) \]
\[-\left( A_\mu^\nu \right)^\rho = - \frac{\partial^2 A_\mu^\nu}{\partial x^\rho \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\rho^\nu \right), \quad (65) \]
\[-\left( A_\mu^\nu \right)^\sigma = - \frac{\partial^2 A_\mu^\nu}{\partial x^\sigma \partial x_\mu} - \frac{\partial A_\mu^\nu}{\partial x_\mu} \frac{1}{2} \left( \partial g_\sigma^\nu \right), \quad (66) \]

By the way, by establishment of \( A_\mu^\nu = A_\nu^\mu \), (58) and (64) must be equal. (58) and (64) are equal in consideration of Definition 5. Furthermore, by establishment of \( A_\mu^\nu = A_\mu^\nu \), (59) and (65) must be equal. (59) and (65) aren’t equal in consideration of Definition 5. Thus, I decide not to handle (59), (65). Furthermore, by establishment of \( A_\mu^\nu = A_\mu^\nu \), (60) and (66) must be equal. (60) and (66) aren’t equal in consideration of Definition 5. Thus, I decide not to handle (60), (66).

Similarly, I rewrite index \( \nu \) of the coordinate systems of (22), (23), (24) by Definition 3 and get
\[
A^\mu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu}}{\partial x^\sigma} \right), \quad (67)
\]
\[
A^\mu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu}}{\partial x^\sigma} \right), \quad (68)
\]
\[
A^\mu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu}}{\partial x^\sigma} \right), \quad (69)
\]

There isn't index \( \nu \) in the second term of the right side of (67), (68), (69) at all here. Therefore, I rewrite dummy index \( \sigma \) of (67), (68), (69) in dummy index \( \nu \) and can get
\[
A^\mu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\nu} \right), \quad (70)
\]
\[
A^\mu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\nu} \right), \quad (71)
\]
\[
A^\mu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\nu} \right), \quad (72)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (22), (23), (24) by Definition 2 and get
\[
A^\mu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu}}{\partial x^\sigma} \right), \quad (73)
\]
\[
A^\mu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\sigma\rho}_{\mu}}{\partial x^{\sigma\rho}} \right), \quad (74)
\]
\[
A^\mu_{\mu,\mu} = \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\sigma\rho}_{\mu}}{\partial x^{\sigma\rho}} \right), \quad (75)
\]

Because two same contravariant index existed in \( A^\mu_{\mu,\mu} \) of (73),(74),(75), I decided not to handle (73),(74),(75). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (22),(23),(24) by Definition 3, Definition 2 and get
\[
-(A^\mu_{\mu})^{ij} = \frac{\partial^2 A^\mu}{\partial x^i \partial x^j} + \frac{\partial A^\mu}{\partial x^i} \frac{1}{2} \left( \frac{\partial g^\sigma_{i}}{\partial x^\sigma} \right), \quad (76)
\]
\[
-(A^\mu_{\mu})^{ij} = \frac{\partial^2 A^\mu}{\partial x^i \partial x^j} + \frac{\partial A^\mu}{\partial x^i} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\nu} \right), \quad (77)
\]
\[
-(A^\mu_{\mu})^{ij} = \frac{\partial^2 A^\mu}{\partial x^i \partial x^j} + \frac{\partial A^\mu}{\partial x^i} \frac{1}{2} \left( \frac{\partial g^\nu_{i}}{\partial x^\nu} \right), \quad (78)
\]

There isn't index \( \nu \) in the second term of the right side of (76),(77),(78) at all here. Therefore, I rewrite dummy index \( \sigma \) of (76),(77),(78) in dummy index \( \nu \) and can get
\[
-(A^\mu_{\mu})^{ij} = \frac{\partial^2 A^\mu}{\partial x^i \partial x^j} + \frac{\partial A^\mu}{\partial x^i} \frac{1}{2} \left( \frac{\partial g^\nu_{}\mu}{\partial x^\nu} \right), \quad (79)
\]
\[
-(A^\mu_{\mu})^{ij} = \frac{\partial^2 A^\mu}{\partial x^i \partial x^j} + \frac{\partial A^\mu}{\partial x^i} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\nu} \right), \quad (80)
\]
\[ -(A_{\mu}^{\nu})_{\mu} = -\frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\mu} \partial x_{\mu}} + \frac{\partial A_{\mu}^{\nu}}{\partial x_{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x_{\mu}} \right) \ldots \quad (81) \]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (22), (23), (24) by Definition 2, Definition 3 and get

\[ -(A^{\mu;\nu})_{\mu} = -\frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\mu} \partial x_{\mu}} + \frac{\partial A_{\mu}^{\nu}}{\partial x_{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x_{\mu}} \right) \ldots \quad (82) \]

\[ -(A_{\mu;\nu})_{\mu} = -\frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\mu} \partial x_{\mu}} + \frac{\partial A_{\mu}^{\nu}}{\partial x_{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x_{\mu}} \right) \ldots \quad (83) \]

\[ -(A^{\mu;\nu})_{\mu} = -\frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\mu} \partial x_{\mu}} + \frac{\partial A_{\mu}^{\nu}}{\partial x_{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x_{\mu}} \right) \ldots \quad (84) \]

There isn’t index \( \nu \) in the second term of the right side of (82), (83), (84) at all here. Therefore, I rewrite dummy index \( \sigma \) of (82), (83), (84) in dummy index \( \nu \) and can get

\[ -(A^{\mu;\nu})_{\mu} = -\frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\mu} \partial x_{\mu}} + \frac{\partial A_{\mu}^{\nu}}{\partial x_{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x_{\mu}} \right) \ldots \quad (85) \]

\[ -(A_{\mu;\nu})_{\mu} = -\frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\mu} \partial x_{\mu}} + \frac{\partial A_{\mu}^{\nu}}{\partial x_{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x_{\mu}} \right) \ldots \quad (86) \]

\[ -(A^{\mu;\nu})_{\mu} = -\frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\mu} \partial x_{\mu}} + \frac{\partial A_{\mu}^{\nu}}{\partial x_{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x_{\mu}} \right) \ldots \quad (87) \]

By the way, by establishment of \( A^{\mu;\nu}_{\nu\nu} = A^{\mu;\nu}_{\nu\nu} \) (79) and (85) must be equal. (79) and (85) are equal in consideration of Definition 5. Furthermore, by establishment of \( A^{\mu;\nu}_{\nu\mu} = A^{\mu;\nu}_{\nu\mu} \) (80) and (86) must be equal. (80) and (86) aren’t equal in consideration of Definition 5. Thus, I decide not to handle (80), (86). Furthermore, by establishment of \( A^{\mu;\nu}_{\nu\mu} = A^{\mu;\nu}_{\nu\mu} \) (81) and (87) must be equal. (81) and (87) aren’t equal in consideration of Definition 5. Thus, I decide not to handle (81), (87). And I get

\[ A^{\mu;\nu}_{\mu\mu} = \frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\mu} \partial x_{\mu}} \quad (88) \]

from (28), (29), (30), (49), (50), (51), (70), (71), (72) in consideration of Definition 5. And I get

\[ -(A_{\mu}^{\mu})_{\mu} = -\frac{\partial^2 A_{\mu}^{\mu}}{\partial x_{\mu} \partial x_{\mu}} \quad (89) \]

from (37), (58), (79) in consideration of Definition 5. And I get

\[ -(A^{\mu;\nu})_{\mu} = -\frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\mu} \partial x_{\mu}} \quad (90) \]

from (43), (64), (85) in consideration of Definition 5. In addition,

\[ A^{\mu;\nu} = \frac{\partial^2 A_{\mu}^{\nu}}{\partial x_{\nu} \partial x_{\nu}} \quad (91) \]

can rewrite (88), (89), (90) using Definition 2, Definition 3 because the second term doesn’t exist in the right side of (88), (89), (90). Because (91) doesn’t change the form of the equation even if all coordinate systems satisfies Binary Law, I get the conclusion that (91) satisfies Binary Law. In other words, (91) is an equation of the tensor satisfying Binary Law. Because (91) is an equation provided from Definition 7, (91) is the equation that was rewritten so that
Definision 7 is an equation of the tensor satisfying Binary Law.

– End Proof –

5 Tensor satisfying Binary Law about the third-order Covariant Derivative of the Contravariant Vector

Proposition 5 Definision 8 isn’t an equation of the tensor satisfying Binary Law.

Proof: If all coordinate systems satisfies Binary Law, I get

\[
A^\mu_{\nu, \rho \gamma} = \frac{\partial^3 A^\mu}{\partial \chi^\nu \partial \chi^\rho \partial \chi^\gamma} \\
+ \frac{\partial^2}{\partial \chi^\nu \partial \chi^\rho} \left( A^\nu \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \right) \\
+ \frac{\partial}{\partial \chi^\nu} \left( A^\nu \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \right) \frac{1}{2} g^{\rho \gamma} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \\
- \frac{\partial}{\partial \chi^\nu} \left( A^\nu \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \right) \frac{1}{2} g^{\rho \gamma} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \\
+ \frac{\partial A^\nu}{\partial \chi^\rho} \left( A^\nu \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \right) \frac{1}{2} g^{\rho \gamma} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \\
+ \frac{\partial A^\nu}{\partial \chi^\nu} \left( A^\nu \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \right) \frac{1}{2} g^{\rho \gamma} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \\
+ \frac{\partial^2 A^\nu}{\partial \chi^\nu \partial \chi^\rho} \left( A^\nu \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \right) \frac{1}{2} g^{\rho \gamma} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \\
+ \frac{\partial^2 A^\nu}{\partial \chi^\nu \partial \chi^\rho} \left( A^\nu \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right) \right) \frac{1}{2} g^{\rho \gamma} \left( \frac{\partial g_{\nu \gamma}}{\partial \chi^\nu} + \frac{\partial g_{\nu \gamma}}{\partial \chi^\rho} - \frac{\partial g_{\nu \gamma}}{\partial \chi^\gamma} \right)
\]
\[-A^{\nu} \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\rho \sigma}}{\partial x^{\nu}} - \frac{\partial g_{\rho \sigma}}{\partial x^{\nu}} \right) \frac{1}{2} g^{\rho \mu} \left( \frac{\partial g_{\sigma \mu}}{\partial x^{\nu}} - \frac{\partial g_{\sigma \mu}}{\partial x^{\nu}} \right) \frac{1}{2} g^{\nu \sigma} \left( \frac{\partial g_{\mu \sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu \sigma}}{\partial x^{\nu}} \right) \]

\[+ A^{\nu} \frac{1}{2} g^{\mu \nu} \left( \frac{\partial g_{\mu \sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu \sigma}}{\partial x^{\nu}} \right) \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\rho \sigma}}{\partial x^{\nu}} - \frac{\partial g_{\rho \sigma}}{\partial x^{\nu}} \right) \frac{1}{2} g^{\nu \sigma} \left( \frac{\partial g_{\sigma \mu}}{\partial x^{\nu}} - \frac{\partial g_{\sigma \mu}}{\partial x^{\nu}} \right) \]

\[-\frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\rho}} \]

\[= -A^{\nu} \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\rho \sigma}}{\partial x^{\nu}} - \frac{\partial g_{\rho \sigma}}{\partial x^{\nu}} \right) \frac{1}{2} g^{\rho \mu} \left( \frac{\partial g_{\sigma \mu}}{\partial x^{\nu}} - \frac{\partial g_{\sigma \mu}}{\partial x^{\nu}} \right) \frac{1}{2} g^{\nu \sigma} \left( \frac{\partial g_{\mu \sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu \sigma}}{\partial x^{\nu}} \right) \]

\[+ A^{\nu} \frac{1}{2} g^{\mu \nu} \left( \frac{\partial g_{\mu \sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu \sigma}}{\partial x^{\nu}} \right) \frac{1}{2} g^{\nu \rho} \left( \frac{\partial g_{\rho \sigma}}{\partial x^{\nu}} - \frac{\partial g_{\rho \sigma}}{\partial x^{\nu}} \right) \frac{1}{2} g^{\nu \sigma} \left( \frac{\partial g_{\sigma \mu}}{\partial x^{\nu}} - \frac{\partial g_{\sigma \mu}}{\partial x^{\nu}} \right) \]
\[+2 \frac{\partial A^\nu}{\partial \xi^\mu} \frac{1}{2} \left( \frac{\partial g^\nu}{\partial \xi^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\nu}{\partial x^\tau} \right) \quad (101)\]

from Definition 8. I rewrite dummy index of (93) in the index which isn’t used in this term and must get

\[A_{\nu,\sigma}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\nu} + \frac{\partial^2}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\nu}{\partial x^\tau} \right) \right) \quad (102)\]

\[A_{\nu,\sigma}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\sigma} + \frac{\partial^2}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\nu}{\partial x^\sigma} \right) \right) \quad (103)\]

\[A_{\nu,\sigma}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\nu} + \frac{\partial^2}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\nu}{\partial x^\nu} \right) \right) \quad (104)\]

\[A_{\nu,\sigma}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\sigma} + \frac{\partial^2}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\nu}{\partial x^\sigma} \right) \right) \quad (105)\]

because (92) must be an equation of the tensor. I get the conclusion that (102), (103), (104), (105) doesn’t satisfy Binary Law from Definition 4. Similarly, I rewrite dummy index of (94) in the index which isn’t used in this term and must get

\[A_{\nu,\nu}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} + \frac{1}{\partial x^\nu} \left( \frac{\partial A^{\sigma}}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\sigma}}{\partial x^\nu} \right) \right) \quad (106)\]

\[A_{\nu,\nu}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\sigma} + \frac{1}{\partial x^\nu} \left( \frac{\partial A^{\sigma}}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\sigma}}{\partial x^\nu} \right) \right) \quad (107)\]

\[A_{\nu,\nu}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\nu} + \frac{1}{\partial x^\nu} \left( \frac{\partial A^{\sigma}}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\sigma}}{\partial x^\nu} \right) \right) \quad (108)\]

\[A_{\nu,\nu}^\mu = \frac{\partial^3 A^\mu}{\partial x^\sigma \partial x^\sigma \partial x^\nu} + \frac{1}{\partial x^\sigma} \left( \frac{\partial A^{\sigma}}{\partial x^\sigma} \frac{1}{2} \left( \frac{\partial g^{\sigma}}{\partial x^\sigma} \right) \right) \quad (109)\]

because (92) must be an equation of the tensor. I get the conclusion that (106), (107), (108), (109) doesn’t satisfy Binary Law from Definition 4. Similarly, I rewrite dummy index of (95) in the index which isn’t used in this term and must get

\[A_{\nu,\nu}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} + \frac{1}{\partial x^\nu} \left( \frac{\partial A^{\sigma}}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\sigma}}{\partial x^\nu} \right) \right) \quad (110)\]

\[A_{\nu,\nu}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\sigma} + \frac{1}{\partial x^\nu} \left( \frac{\partial A^{\sigma}}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\sigma}}{\partial x^\nu} \right) \right) \quad (111)\]

\[A_{\nu,\nu}^\mu = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\nu} + \frac{1}{\partial x^\nu} \left( \frac{\partial A^{\sigma}}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\sigma}}{\partial x^\nu} \right) \right) \quad (112)\]

\[A_{\nu,\nu}^\mu = \frac{\partial^3 A^\mu}{\partial x^\sigma \partial x^\sigma \partial x^\nu} + \frac{1}{\partial x^\sigma} \left( \frac{\partial A^{\sigma}}{\partial x^\sigma} \frac{1}{2} \left( \frac{\partial g^{\sigma}}{\partial x^\sigma} \right) \right) \quad (113)\]

because (92) must be an equation of the tensor. I get the conclusion that (110), (111), (112), (113) doesn’t satisfy Binary Law from Definition 4. Similarly, I rewrite dummy index of (96) in the index which isn’t used in this term and must get
because (92) must be an equation of the tensor. I get the conclusion that (114), (115), (116), (117) doesn’t satisfy Binary Law from Definition 4. Similarly, I rewrite dummy index of (97) in the index which isn’t used in this term and must get

\[
A_{\nu,y,y}^{\mu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\mu}^{\sigma}}{\partial \chi} \right), \quad (114)
\]

\[
A_{\nu,y,y}^{\mu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (115)
\]

\[
A_{\nu,y,y}^{\mu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (116)
\]

\[
A_{\nu,y,y}^{\mu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (117)
\]

because (92) must be an equation of the tensor. I get the conclusion that (118) ~ (127) doesn’t satisfy Binary Law from Definition 4. Similarly, I rewrite dummy index of (97) in the index which isn’t used in this term and must get

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\mu}^{\sigma}}{\partial \chi} \right), \quad (118)
\]

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (119)
\]

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (120)
\]

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (121)
\]

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (122)
\]

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (123)
\]

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (124)
\]

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (125)
\]

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (126)
\]

\[
A_{\mu,\mu,\mu}^{\nu} = \partial_{\chi} \left( \frac{\partial^{3} A^{\mu}}{\partial \chi \partial \chi' \partial \chi''} \right) + \frac{1}{2} \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right) \left( \frac{\partial g_{\nu}^{\sigma}}{\partial \chi} \right), \quad (127)
\]
\[ A_{\mu,\nu,\rho}^{\sigma} = \frac{\partial A_{\mu}}{\partial x^{\nu}} \frac{\partial A_{\rho}}{\partial x^{\sigma}} - 2 \frac{\partial A_{\mu}}{\partial x^{\nu}} \left( \frac{1}{2} \frac{\partial g_{\sigma}}{\partial x^{\nu}} \right)^{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\rho}} \right), \]  \hspace{1cm} (129)

\[ A_{\mu,\nu,\rho}^{\sigma} = \frac{\partial A_{\mu}}{\partial x^{\nu}} \frac{\partial A_{\rho}}{\partial x^{\sigma}} - 2 \frac{\partial A_{\mu}}{\partial x^{\nu}} \left( \frac{1}{2} \frac{\partial g_{\sigma}}{\partial x^{\nu}} \right)^{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\rho}} \right), \]  \hspace{1cm} (130)

\[ A_{\mu,\nu,\rho}^{\sigma} = \frac{\partial A_{\mu}}{\partial x^{\nu}} \frac{\partial A_{\rho}}{\partial x^{\sigma}} - 2 \frac{\partial A_{\mu}}{\partial x^{\nu}} \left( \frac{1}{2} \frac{\partial g_{\sigma}}{\partial x^{\nu}} \right)^{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\rho}} \right), \]  \hspace{1cm} (131)

because (92) must be an equation of the tensor. I get the conclusion that (128) \( \sim (137) \) doesn’t satisfy Binary Law from Definision 4. Similarly, I rewrite dummy index of (99) in the index which isn’t used in this term and must get

\[ A_{\mu,\nu,\rho}^{\sigma} = - \frac{\partial^{2} A_{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\nu}} \right)^{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\rho}} \right), \]  \hspace{1cm} (138)

\[ A_{\mu,\nu,\rho}^{\sigma} = - \frac{\partial^{2} A_{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\nu}} \right)^{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\rho}} \right), \]  \hspace{1cm} (139)

\[ A_{\mu,\nu,\rho}^{\sigma} = - \frac{\partial^{2} A_{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\nu}} \right)^{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\rho}} \right), \]  \hspace{1cm} (140)

because (92) must be an equation of the tensor. I get the conclusion that (138), (139), (140), (141) doesn’t satisfy Binary Law from Definision 4. Similarly, I rewrite dummy index of (100) in the index which isn’t used in this term and must get

\[ A_{\mu,\nu,\rho}^{\sigma} = - \frac{\partial^{2} A_{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\nu}} \right)^{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\rho}} \right), \]  \hspace{1cm} (142)

\[ A_{\mu,\nu,\rho}^{\sigma} = - \frac{\partial^{2} A_{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\nu}} \right)^{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\rho}} \right), \]  \hspace{1cm} (143)
because (92) must be an equation of the tensor. I get the conclusion that (142) \sim (151) doesn’t satisfy Binary Law from Definition 4. Similarly, I rewrite dummy index of (101) in the index which isn’t used in this term and must get

\[ A^\mu_{\nu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\nu_{\sigma\sigma}}{\partial x^\nu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\sigma_{\sigma\sigma}}{\partial x^\nu} \right), \quad (144) \]

\[ A^\mu_{\nu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\nu_{\sigma\sigma}}{\partial x^\nu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\sigma_{\sigma\sigma}}{\partial x^\nu} \right), \quad (145) \]

\[ A^\mu_{\nu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\nu_{\sigma\sigma}}{\partial x^\nu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\sigma_{\sigma\sigma}}{\partial x^\nu} \right), \quad (146) \]

\[ A^\mu_{\nu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\nu_{\sigma\sigma}}{\partial x^\nu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\sigma_{\sigma\sigma}}{\partial x^\nu} \right), \quad (147) \]

\[ A^\mu_{\nu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\nu_{\sigma\sigma}}{\partial x^\nu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\sigma_{\sigma\sigma}}{\partial x^\nu} \right), \quad (148) \]

\[ A^\mu_{\nu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\nu_{\sigma\sigma}}{\partial x^\nu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\sigma_{\sigma\sigma}}{\partial x^\nu} \right), \quad (149) \]

\[ A^\mu_{\nu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\nu_{\sigma\sigma}}{\partial x^\nu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\sigma_{\sigma\sigma}}{\partial x^\nu} \right), \quad (150) \]

\[ A^\mu_{\nu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\nu_{\sigma\sigma}}{\partial x^\nu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\sigma_{\sigma\sigma}}{\partial x^\nu} \right), \quad (151) \]
because (92) must be an equation of the tensor. I get the conclusion that (152) ~ (161) doesn’t satisfy Binary Law from Definition 4. If (102) ~ (161) satisfies Binary Law, Definition 8 is an equation of the tensor satisfying Binary Law here. However, I get the conclusion that Definition 8 isn’t an equation of the tensor satisfying Binary Law because (102) ~ (161) doesn’t satisfy Binary Law.

– End Proof –

Proposition 6 $A_{\nu;\rho;\tau;\gamma}^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho \partial x^\tau} \cdots$ is the equation that was rewritten so that Definition 8 is an equation of the tensor satisfying Binary Law.

Proof: I rewrite index $\nu$ of the coordinate systems of (102), (103), (104), (105) by Definition 3 and get

\begin{equation}
-A_{\mu;\nu;\rho;\mu}^{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\mu \partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\rho;\sigma}}{\partial x^\sigma} \right) \right) \cdots, \tag{162}
\end{equation}

\begin{equation}
-A_{\mu;\nu;\mu;\nu}^{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\nu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\nu;\sigma}}{\partial x^\sigma} \right) \right) \cdots, \tag{163}
\end{equation}

\begin{equation}
-A_{\mu;\nu;\nu;\rho}^{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\nu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\nu;\sigma}}{\partial x^\sigma} \right) \right) \cdots, \tag{164}
\end{equation}

\begin{equation}
-A_{\mu;\rho;\mu;\nu}^{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\rho \partial x^\nu} + \frac{\partial^2}{\partial x^\mu \partial x^\rho} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\mu;\sigma}}{\partial x^\sigma} \right) \right) \cdots. \tag{165}
\end{equation}

Because two same covariant index existed in $-A_{\mu;\nu;\rho;\mu}^{\mu}$ of (162), (163), (164), (165), I decided not to handle (162), (163), (164), (165). Furthermore, I rewrite index $\nu$ of the coordinate systems of (102), (103), (104), (105) by Definition 2 and get

\begin{equation}
A_{\mu;\nu;\rho;\mu}^{\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\nu \partial x_\rho} + \frac{\partial^2}{\partial x_\mu \partial x_\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\rho;\sigma}}{\partial x_\sigma} \right) \right) \cdots, \tag{166}
\end{equation}

\begin{equation}
A_{\mu;\nu;\mu;\nu}^{\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\nu \partial x_\nu} + \frac{\partial^2}{\partial x_\mu \partial x_\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\nu;\sigma}}{\partial x_\sigma} \right) \right) \cdots, \tag{167}
\end{equation}

\begin{equation}
A_{\mu;\nu;\nu;\rho}^{\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\nu \partial x_\nu} + \frac{\partial^2}{\partial x_\mu \partial x_\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\nu;\sigma}}{\partial x_\sigma} \right) \right) \cdots, \tag{168}
\end{equation}

\begin{equation}
A_{\mu;\rho;\mu;\nu}^{\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\rho \partial x_\nu} + \frac{\partial^2}{\partial x_\mu \partial x_\rho} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\mu;\sigma}}{\partial x_\sigma} \right) \right) \cdots. \tag{169}
\end{equation}

Because two same contravariant index existed in $A_{\mu;\nu;\rho;\mu}^{\mu}$ of (166), (167), (168), (169), I decided not to handle (166), (167), (168), (169). Furthermore, I rewrite index $\nu$ of the coordinate systems of (102), (103), (104), (105) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (170)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (171)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (172)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (173)
\]

There isn’t index \( V \) in the second term of the right side of (170), (171), (172), (173) at all here. Therefore, I rewrite dummy index \( \sigma \) of (170), (171), (172), (173) in dummy index \( \nu \) and can get
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (174)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (175)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (176)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (177)
\]

Furthermore, I rewrite index \( V \) of the coordinate systems of (102), (103), (104), (105) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (178)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (179)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (180)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (181)
\]

There isn’t index \( V \) in the second term of the right side of (178), (179), (180), (181) at all here. Therefore, I rewrite dummy index \( \sigma \) of (178), (179), (180), (181) in dummy index \( \nu \) and can get
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (182)
\]
\[
\left(A^\mu_{\nu,\rho}\right)^\gamma = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial^2}{\partial x^\rho} \left( A^\rho \frac{1}{2} \frac{\partial g^\mu_{\nu,\sigma}}{\partial x^\sigma} \right), \quad (183)
\]
\[
\left( A^\mu_{\nu\mu} \right)^\nu_{,\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (184)
\]
\[
\left( A^\mu_{\nu\mu} \right)^\mu_{,\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial^2}{\partial x^\mu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (185)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (102), (103), (104), (105) by Definition 3, Definition 2 and get

\[
\left( A^\mu_{\nu\mu} \right)^\mu_{,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\mu} + \frac{\partial^2}{\partial x^\mu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (186)
\]
\[
\left( A^\mu_{\nu\mu} \right)^\nu_{,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (187)
\]
\[
\left( A^\mu_{\nu\mu} \right)^\nu_{,\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (188)
\]
\[
\left( A^\mu_{\nu\mu} \right)^\nu_{,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\mu} + \frac{\partial^2}{\partial x^\mu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (189)
\]

There isn’t index \( \nu \) in the second term of the right side of (186), (187), (188), (189) at all here. Therefore, I rewrite dummy index \( \sigma \) of (186), (187), (188), (189) in dummy index \( \nu \) and can get

\[
\left( A^\mu_{\nu\mu} \right)^\nu_{,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (190)
\]
\[
\left( A^\mu_{\nu\mu} \right)^\nu_{,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (191)
\]
\[
\left( A^\mu_{\nu\mu} \right)^\nu_{,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (192)
\]
\[
\left( A^\mu_{\nu\mu} \right)^\nu_{,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (193)
\]

By the way, by establishment of \( A^\mu_{\nu\mu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} \), (174), (182), (190) are equal each other. (174), (182), (190) are equal each other in consideration of Definition 5. Furthermore, by establishment of \( A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} \), (175), (183), (191) must be equal each other. (175), (183), (191) aren’t equal each other in consideration of Definition 5. Thus, I decide not to handle (175), (183), (191). Furthermore, by establishment of \( A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} \), (176), (184), (192) are equal each other. (176), (184), (192) aren’t equal each other in consideration of Definition 5. Thus, I decide not to handle (176), (184), (192). Furthermore, by establishment of \( A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} = A^\mu_{\nu\nu} \), (177), (185), (193) must be equal each other. (177), (185), (193) aren’t equal each other in consideration of Definition 5. Thus, I decide not to handle (177), (185), (193). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (102), (103), (104), (105) by Definition 3, Definition 2 and get

\[
-(A^\mu_{\nu\mu})^\mu_{,\nu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\mu} - \frac{\partial^2}{\partial x^\mu \partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \right) \ldots, (194)
\]
Because two same contravariant index existed in $-(A^\mu)_{;\mu}^{\mu;\mu}$ of (194), (195), (196), (197), I decided not to handle (194), (195), (196), (197). Furthermore, I rewrite index $\nu$ of the coordinate systems of (102), (103), (104), (105) by Definition 3, Definition 2 and get

\[-(A_\mu^{\mu,\mu})_{;\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu^3} A^\mu - \frac{\partial^2}{\partial x_\mu^2} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x_\mu} \right) \right) \ldots, \quad (198)\]

\[-(A^{\mu,\mu,\mu})_{;\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu^3} A^\mu - \frac{\partial^2}{\partial x_\mu^2} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\mu}}{\partial x_\mu} \right) \right) \ldots, \quad (199)\]

\[-(A^{\mu,\mu,\mu})_{;\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu^3} A^\mu - \frac{\partial^2}{\partial x_\mu^2} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\mu}}{\partial x_\mu} \right) \right) \ldots, \quad (200)\]

\[-(A^{\mu,\mu,\mu})_{;\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu^3} A^\mu - \frac{\partial^2}{\partial x_\mu^2} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\mu}}{\partial x_\mu} \right) \right) \ldots, \quad (201)\]

Because two same contravariant index existed in $-(A^{\mu,\mu,\mu})_{;\mu}$ of (198), (199), (200), (201), I decided not to handle (198), (199), (200), (201). Furthermore, I rewrite index $\nu$ of the coordinate systems of (106), (107), (108), (109) by Definition 3, Definition 2 and get

\[-(A^{\mu,\mu,\mu})_{;\nu} = -\frac{\partial^3 A^\mu}{\partial x_\mu^3} A^\nu - \frac{\partial^2}{\partial x_\mu^2} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x_\nu} \right) \right) \ldots, \quad (202)\]

\[-(A^{\mu,\mu,\mu})_{;\nu} = -\frac{\partial^3 A^\mu}{\partial x_\mu^3} A^\nu - \frac{\partial^2}{\partial x_\mu^2} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x_\mu} \right) \right) \ldots, \quad (203)\]

\[-(A^{\mu,\mu,\mu})_{;\nu} = -\frac{\partial^3 A^\mu}{\partial x_\mu^3} A^\nu - \frac{\partial^2}{\partial x_\mu^2} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x_\mu} \right) \right) \ldots, \quad (204)\]

\[-(A^{\mu,\mu,\mu})_{;\nu} = -\frac{\partial^3 A^\mu}{\partial x_\mu^3} A^\nu - \frac{\partial^2}{\partial x_\mu^2} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x_\mu} \right) \right) \ldots, \quad (205)\]

Because two same contravariant index existed in $-(A^{\mu,\mu,\mu})_{;\mu}$ of (202), (203), (204), (205), I decided not to handle (202), (203), (204), (205). Similarly, I rewrite index $\nu$ of the coordinate systems of (106), (107), (108), (109) by Definition 3 and get

\[-A^{\mu,\mu,\mu}_{;\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu^3} A^\nu - \frac{\partial^2}{\partial x_\mu^2} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x_\nu} \right) \right) \ldots, \quad (206)\]
\[- A^\mu_{\nu,\mu;\nu} = - \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (207)

\[- A^\nu_{\mu,\mu;\nu} = - \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (208)

\[- A^\mu_{\nu,\mu;\nu} = - \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (209)

Because two same covariant index existed in \(- A^\mu_{\nu,\mu;\nu}\) of (206), (207), (208), (209), I decided not to handle (206), (207), (208), (209). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (106), (107), (108), (109) by Definition 2 and get

\[ A^\mu_{\nu,\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (210)

\[ A^\nu_{\mu,\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (211)

\[ A^\mu_{\nu,\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (212)

\[ A^\mu_{\nu,\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (213)

Because two same contravariant index existed in \( A^\mu_{\nu,\mu;\nu}\) of (210), (211), (212), (213), I decided not to handle (210), (211), (212), (213). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (106), (107), (108), (109) by Definition 3, Definition 2 and get

\[ \left( A^\mu_{\nu,\mu;\nu} \right)^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (214)

\[ \left( A^\mu_{\nu,\mu;\nu} \right)^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (215)

\[ \left( A^\mu_{\nu,\mu;\nu} \right)^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (216)

\[ \left( A^\mu_{\nu,\mu;\nu} \right)^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (217)

There isn’t index \(\nu\) in the second term of the right side of (214), (215), (216), (217) at all here. Therefore, I rewrite dummy index \(\sigma\) of (214), (215), (216), (217) in dummy index \(\nu\) and can get

\[ \left( A^\mu_{\nu,\mu;\nu} \right)^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (218)

\[ \left( A^\mu_{\nu,\mu;\nu} \right)^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_{\nu}}{\partial x^\sigma} \right) \right) \ldots \] (219)
\[
\left( A_{\mu,\nu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{220}
\]

\[
\left( A_{\mu,\nu}^{\sigma} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{221}
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (106), (107), (108), (109) by Definition 3, Definition 2 and get

\[
\left( A_{\mu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\sigma}{\partial x^{\nu}} \right) \right), \tag{222}
\]

\[
\left( A_{\mu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{223}
\]

\[
\left( A_{\mu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\sigma}{\partial x^{\nu}} \right) \right), \tag{224}
\]

\[
\left( A_{\mu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\sigma}{\partial x^{\nu}} \right) \right), \tag{225}
\]

There isn’t index \( \nu \) in the second term of the right side of (222), (223), (224), (225) at all here. Therefore, I rewrite dummy index \( \sigma \) of (222), (223), (224), (225) in dummy index \( \nu \) and can get

\[
\left( A_{\mu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{226}
\]

\[
\left( A_{\mu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{227}
\]

\[
\left( A_{\mu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{228}
\]

\[
\left( A_{\mu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{229}
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (106), (107), (108), (109) by Definition 3, Definition 2 and get

\[
\left( A_{\mu,\nu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{230}
\]

\[
\left( A_{\mu,\nu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{231}
\]

\[
\left( A_{\mu,\nu}^{\alpha} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \frac{1}{2} \left( \frac{\partial g_{\mu}^\nu}{\partial x^{\mu}} \right) \right), \tag{232}
\]
\[
\left( A^{\mu \nu} \right)_{\mu \nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\nu} + \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\sigma}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \nu}}{\partial x^\sigma} \right) \right) \] ...
(233)

There isn’t index \( \nu \) in the second term of the right side of (230), (231), (232), (233) at all here. Therefore, I rewrite dummy index \( \sigma \) of (230), (231), (232), (233) in dummy index \( \nu \) and can get

\[
\left( A^{\mu \nu} \right)_{\mu \nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\nu} + \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\nu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \nu}}{\partial x^\nu} \right) \right) \] ...
(234)

\[
\left( A^{\mu \nu} \right)_{\mu \nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\nu} + \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\nu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \nu}}{\partial x^\nu} \right) \right) \] ...
(235)

\[
\left( A^{\mu \nu} \right)_{\mu \nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\nu} + \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\nu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \nu}}{\partial x^\nu} \right) \right) \] ...
(236)

\[
\left( A^{\mu \nu} \right)_{\mu \nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\nu} + \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\nu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \nu}}{\partial x^\nu} \right) \right) \] ...
(237)

By the way, by establishment of \( A^\mu_{\nu \rho \sigma} = A^\mu_{\rho \nu \sigma} = A^\mu_{\nu \sigma \rho} = A^\mu_{\sigma \nu \rho} = A^\mu_{\rho \sigma \nu} = A^\mu_{\sigma \rho \nu} \), (218), (226), (234) must be equal each other. (218), (226), (234) are equal each other in consideration of Definision 5. Furthermore, by establishment of \( A^\mu_{\nu \rho \sigma} = A^\mu_{\rho \nu \sigma} = A^\mu_{\nu \sigma \rho} = A^\mu_{\sigma \nu \rho} = A^\mu_{\rho \sigma \nu} = A^\mu_{\sigma \rho \nu} \), (219), (227), (235) must be equal each other. (219), (227), (235) aren’t equal each other in consideration of Definision5. Thus, I decide not to handle (219), (227), (235). Furthermore, by establishment of \( A^\mu_{\nu \rho \sigma} = A^\mu_{\rho \nu \sigma} = A^\mu_{\nu \sigma \rho} = A^\mu_{\sigma \nu \rho} = A^\mu_{\rho \sigma \nu} = A^\mu_{\sigma \rho \nu} \), (220), (228), (236) aren’t equal each other in consideration of Definision 5. Thus, I decide not to handle (220), (228), (236). Furthermore, by establishment of \( A^\mu_{\nu \rho \sigma} = A^\mu_{\rho \nu \sigma} = A^\mu_{\nu \sigma \rho} = A^\mu_{\sigma \nu \rho} = A^\mu_{\rho \sigma \nu} = A^\mu_{\sigma \rho \nu} \), (221), (229), (237) must be equal each other. (221), (229), (237) aren’t equal each other in consideration of Definision 5. Thus, I decide not to handle (221), (229), (237). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (106), (107), (108), (109) by Definision 3, Definision 2 and get

\[
-(A^\mu_{\nu \rho \sigma})^\mu_{\nu \rho \sigma} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\rho \partial x^\sigma} - \frac{\partial}{\partial x^\sigma} \left( \frac{\partial A^\rho}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \rho}}{\partial x^\sigma} \right) \right) \] ...
(238)

\[
-(A^\mu_{\nu \rho \sigma})^\mu_{\nu \rho \sigma} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\rho \partial x^\sigma} - \frac{\partial}{\partial x^\sigma} \left( \frac{\partial A^\rho}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \rho}}{\partial x^\sigma} \right) \right) \] ...
(239)

\[
-(A^\mu_{\nu \rho \sigma})^\mu_{\nu \rho \sigma} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\rho \partial x^\sigma} - \frac{\partial}{\partial x^\sigma} \left( \frac{\partial A^\rho}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \rho}}{\partial x^\sigma} \right) \right) \] ...
(240)

\[
-(A^\mu_{\nu \rho \sigma})^\mu_{\nu \rho \sigma} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\rho \partial x^\sigma} - \frac{\partial}{\partial x^\sigma} \left( \frac{\partial A^\rho}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \rho}}{\partial x^\sigma} \right) \right) \] ...
(241)

Because two same contravariant index existed in \(-(A^\mu_{\nu \rho \sigma})^\mu_{\nu \rho \sigma}\) of (238), (239), (240), (241), I decided not to handle (238), (239), (240), (241). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (106), (107), (108), (109) by Definision 3, Definision 2 and get

\[
-(A^\mu_{\nu \rho \sigma})^\mu_{\nu \rho \sigma} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\rho \partial x^\rho} - \frac{\partial}{\partial x^\rho} \left( \frac{\partial A^\rho}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g_{\mu \rho}}{\partial x^\rho} \right) \right) \] ...
(242)
\[ -\left( A^{\mu,\nu,\rho}_{\mu,\nu,\rho} \right)_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (243) \]
\[ -\left( A^{\mu,\nu,\rho}_{\mu,\nu,\rho} \right)_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (244) \]
\[ -\left( A^{\mu,\nu,\rho}_{\mu,\nu,\rho} \right)_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (245) \]

Because two same contravariant index existed in \(-\left( A^{\mu,\nu,\rho}_{\mu,\nu,\rho} \right)_{\mu}\) of (242), (243), (244), (245), I decided not to handle (242), (243), (244), (245). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (105) by Definition 3 and get

\[ -\left( A^{\mu,\nu,\rho}_{\mu,\nu,\rho} \right)_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (246) \]
\[ -\left( A^{\mu,\nu,\rho}_{\mu,\nu,\rho} \right)_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (247) \]
\[ -\left( A^{\mu,\nu,\rho}_{\mu,\nu,\rho} \right)_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (248) \]
\[ -\left( A^{\mu,\nu,\rho}_{\mu,\nu,\rho} \right)_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (249) \]

Because two same contravariant index existed in \(-\left( A^{\mu,\nu,\rho}_{\mu,\nu,\rho} \right)_{\mu}\) of (246), (247), (248), (249), I decided not to handle (246), (247), (248), (249). Similarly, I rewrite index \(\nu\) of the coordinate systems of (110) by Definition 3 and get

\[ -A^{\mu,\nu,\rho}_{\mu,\nu,\rho} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (250) \]
\[ -A^{\mu,\nu,\rho}_{\mu,\nu,\rho} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (251) \]
\[ -A^{\mu,\nu,\rho}_{\mu,\nu,\rho} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (252) \]
\[ -A^{\mu,\nu,\rho}_{\mu,\nu,\rho} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (253) \]

Because two same covariant index existed in \(-A^{\mu,\nu,\rho}_{\mu,\nu,\rho}\) of (250), (251), (252), (253), I decided not to handle (250), (251), (252), (253). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (110) by Definition 2 and get

\[ A^{\mu,\nu,\rho}_{\mu,\nu,\rho} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (254) \]
\[ A^{\mu,\nu,\rho}_{\mu,\nu,\rho} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\sigma}{\partial x^\mu} - \frac{1}{2} \frac{\partial^2 g^\mu}{\partial x^\mu} \right), \quad (255) \]
Because two same contravariant index existed in $A_{\mu;\mu;\mu;\mu}$ of (254), (255), (256), (257), I decided not to handle (254), (255), (256), (257). Furthermore, I rewrite index $\nu$ of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma\sigma}}{\partial x^{\mu}} \right) \right), \quad (258)
$$

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma\sigma}}{\partial x^{\mu}} \right) \right), \quad (259)
$$

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma\sigma}}{\partial x^{\mu}} \right) \right), \quad (260)
$$

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma\sigma}}{\partial x^{\mu}} \right) \right), \quad (261)
$$

There isn’t index $\nu$ in the second term of the right side of (258), (259), (260), (261) at all here. Therefore, I rewrite dummy index $\sigma$ of (258), (259), (260), (261) in dummy index $\nu$ and can get

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu\nu}}{\partial x^{\mu}} \right) \right), \quad (262)
$$

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu\nu}}{\partial x^{\mu}} \right) \right), \quad (263)
$$

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu\nu}}{\partial x^{\mu}} \right) \right), \quad (264)
$$

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu\nu}}{\partial x^{\mu}} \right) \right), \quad (265)
$$

Furthermore, I rewrite index $\nu$ of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} \right) \right), \quad (266)
$$

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} \right) \right), \quad (267)
$$

$$
\left( A_{\mu;\mu;\mu;\mu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial A^{\mu}}{\partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} \right) \right), \quad (268)
\[
\left(A^\mu_{\nu}\right)_{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\sigma} - \frac{\partial}{\partial x^\sigma} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (269)
\]

There isn’t index \( \nu \) in the second term of the right side of (266), (267), (268), (269) at all here. Therefore, I rewrite dummy index \( \sigma \) of (266), (267), (268), (269) in dummy index \( \nu \) and can get
\[
\left(A^\mu_{\nu}\right)_{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (270)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (271)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (272)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (273)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (274)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (275)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (276)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (277)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (278)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (279)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (280)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get
\[
\left(A^\mu_{\nu}\right)_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\nu_{\mu}}{\partial x^\mu} \right) \right) \quad \ldots \quad (281)
\]

By the way, by establishment of \( A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = \ldots \), (262), (270), (278) must be equal each other. (262), (270), (278) are equal each other in consideration of Definition 5. Furthermore, by establishment of \( A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = A^\mu_{\nu;\nu} = \ldots \), (263), (271), (279) must be equal each other. (263), (271), (279) aren’t
equal each other in consideration of Definition 5. Thus, I decide not to handle (263), (271), (279). Furthermore, by establishment of \( A_{\nu,\mu}^{\rho} = A_{\nu,\rho}^{\mu} = A_{\rho,\mu}^{\nu} = A_{\rho,\nu}^{\mu} = A_{\mu,\rho}^{\nu} = A_{\mu,\nu}^{\rho} \), (264), (272), (280) must be equal each other. (264), (272), (280) aren’t equal each other in consideration of Definition 5. Thus, I decide not to handle (264), (272), (280). Furthermore, by establishment of \( A_{\nu,\mu}^{\rho} = A_{\nu,\rho}^{\mu} = A_{\rho,\mu}^{\nu} = A_{\rho,\nu}^{\mu} = A_{\mu,\rho}^{\nu} = A_{\mu,\nu}^{\rho} \), (265), (273), (281) must be equal each other. (265), (273), (281) aren’t equal each other in consideration of Definition 5. Thus, I decide not to handle (265), (273), (281). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get

\[
-(A_{\mu}^{\rho})^{\nu,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{282}
\]

\[
-(A_{\mu}^{\rho})^{\nu,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{283}
\]

\[
-(A_{\mu}^{\rho})^{\nu,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{284}
\]

\[
-(A_{\mu}^{\rho})^{\nu,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{285}
\]

Because two same contravariant index existed in \(-(A_{\mu}^{\rho})^{\nu,\mu}\) of (282), (283), (284), (285), I decided not to handle (282), (283), (284), (285). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get

\[
-(A_{\mu,\mu}^{\rho})^{\nu}_{,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{286}
\]

\[
-(A_{\mu,\mu}^{\rho})^{\nu}_{,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{287}
\]

\[
-(A_{\mu,\mu}^{\rho})^{\nu}_{,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{288}
\]

\[
-(A_{\mu,\mu}^{\rho})^{\nu}_{,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{289}
\]

Because two same contravariant index existed in \(-(A_{\mu,\mu}^{\rho})^{\nu}_{,\mu}\) of (286), (287), (288), (289), I decided not to handle (286), (287), (288), (289). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (110), (111), (112), (113) by Definition 3, Definition 2 and get

\[
-(A_{\mu}^{\rho})^{\nu}_{,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{290}
\]

\[
-(A_{\mu}^{\rho})^{\nu}_{,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{291}
\]

\[
-(A_{\mu}^{\rho})^{\nu}_{,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{292}
\]

\[
-(A_{\mu}^{\rho})^{\nu}_{,\mu} = -\frac{\partial^3 A_{\mu}^{\rho}}{\partial x_\rho \partial x_\mu \partial x_\nu} + \frac{\partial}{\partial x_\mu} \left( \frac{\partial A_{\mu}^{\rho}}{\partial x_\nu} \frac{1}{2} \frac{\partial g_{\sigma}^{\nu}}{\partial x_\mu} \right), \tag{293}
\]
Because two same contravariant index existed in \(-\left( A^{\mu,\nu}_{\mu,\nu,\mu} \right)^{\mu}_{\mu}\) of (290), (291), (292), (293), I decided not to handle (290), (291), (292), (293).

Similarly, I rewrite index \(\nu\) of the coordinate systems of (114), (115), (116), (117) by Definition 3 and get

\[
-A^{\mu,\nu}_{\mu,\nu,\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_{\mu}}{\partial x^\mu}, \tag{294}
\]

\[
-A^{\mu,\nu}_{\mu,\nu,\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_{\mu}}{\partial x^\mu}, \tag{295}
\]

\[
-A^{\mu,\nu}_{\mu,\nu,\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_{\mu}}{\partial x^\mu}, \tag{296}
\]

\[
-A^{\mu,\nu}_{\mu,\nu,\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_{\mu}}{\partial x^\mu}, \tag{297}
\]

Because two same covariant index existed in \(-A^{\mu,\nu}_{\mu,\nu,\mu}\) of (294), (295), (296), (297), I decided not to handle (294), (295), (296), (297). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (114), (115), (116), (117) by Definition 2 and get

\[
A^{\mu,\mu,\mu}_{\nu,\nu,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_{\nu}}{\partial x^\mu}, \tag{298}
\]

\[
A^{\mu,\nu,\mu}_{\nu,\nu,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_{\nu}}{\partial x^\mu}, \tag{299}
\]

\[
A^{\mu,\mu,\mu}_{\nu,\nu,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_{\nu}}{\partial x^\mu}, \tag{300}
\]

\[
A^{\mu,\nu,\mu}_{\nu,\nu,\nu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_{\nu}}{\partial x^\mu}, \tag{301}
\]

Because two same contravariant index existed in \(A^{\mu,\mu,\mu}_{\nu,\nu,\nu}\) of (298), (299), (300), (301), I decided not to handle (298), (299), (300), (301). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (114), (115), (116), (117) by Definition 3, Definition 2 and get

\[
\left( A^{\mu,\mu,\mu}_{\nu,\nu,\nu} \right)^{\mu}_{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu} \frac{\partial^2 A^\mu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_{\nu}}{\partial x^\mu}, \tag{302}
\]

\[
\left( A^{\mu,\nu,\mu}_{\nu,\nu,\nu} \right)^{\mu}_{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_{\nu}}{\partial x^\mu}, \tag{303}
\]

\[
\left( A^{\mu,\mu,\mu}_{\nu,\nu,\nu} \right)^{\mu}_{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_{\nu}}{\partial x^\mu}, \tag{304}
\]

\[
\left( A^{\mu,\nu,\mu}_{\nu,\nu,\nu} \right)^{\mu}_{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} \frac{\partial^2 A^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_{\nu}}{\partial x^\mu}, \tag{305}
\]

There isn’t index \(\nu\) in the second term of the right side of (302), (303), (304), (305) at all here. Therefore, I rewrite dummy index \(\sigma\) of (302), (303), (304), (305) in dummy index \(\nu\) and can get

\[
\left( A^{\mu,\mu,\mu}_{\nu,\nu,\nu} \right)^{\mu}_{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu} \frac{\partial^2 A^\mu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_{\sigma}}{\partial x^\mu}, \tag{306}
\]
Furthermore, I rewrite index $\nu$ of the coordinate systems of (114), (115), (116), (117) by Definition 3, Definition 2 and get

\[
\left( A_{\mu;\nu}^{\rho} \right)_{\mu}^{\nu} = \partial_{\mu} A_{\rho}^{\nu} + \partial_{\nu} A_{\mu}^{\rho} \left( \partial_{\rho} g_{\mu}^{\nu} / \partial_{\mu} \right), \quad (307)
\]

\[
\left( A_{\mu;\nu}^{\rho} \right)_{\mu}^{\nu} = \partial_{\mu} A_{\rho}^{\nu} + \partial_{\nu} A_{\mu}^{\rho} \left( \partial_{\rho} g_{\mu}^{\nu} / \partial_{\mu} \right), \quad (308)
\]

\[
\left( A_{\mu;\nu}^{\rho} \right)_{\mu}^{\nu} = \partial_{\mu} A_{\rho}^{\nu} + \partial_{\nu} A_{\mu}^{\rho} \left( \partial_{\rho} g_{\mu}^{\nu} / \partial_{\mu} \right), \quad (309)
\]

There isn’t index $\nu$ in the second term of the right side of (310), (311), (312), (313) at all here. Therefore, I rewrite dummy index $\sigma$ of (310), (311), (312), (313) in dummy index $\nu$ and can get

\[
\left( A_{\mu;\rho}^{\nu} \right)_{\mu}^{\nu} = \partial_{\mu} A_{\rho}^{\nu} + \partial_{\nu} A_{\mu}^{\rho} \left( \partial_{\rho} g_{\mu}^{\nu} / \partial_{\mu} \right), \quad (314)
\]

\[
\left( A_{\mu;\rho}^{\nu} \right)_{\mu}^{\nu} = \partial_{\mu} A_{\rho}^{\nu} + \partial_{\nu} A_{\mu}^{\rho} \left( \partial_{\rho} g_{\mu}^{\nu} / \partial_{\mu} \right), \quad (315)
\]

\[
\left( A_{\mu;\rho}^{\nu} \right)_{\mu}^{\nu} = \partial_{\mu} A_{\rho}^{\nu} + \partial_{\nu} A_{\mu}^{\rho} \left( \partial_{\rho} g_{\mu}^{\nu} / \partial_{\mu} \right), \quad (316)
\]

Furthermore, I rewrite index $\nu$ of the coordinate systems of (114), (115), (116), (117) by Definition 3, Definition 2 and get

\[
\left( A_{\nu;\mu}^{\rho} \right)_{\nu}^{\mu} = \partial_{\nu} A_{\rho}^{\mu} + \partial_{\mu} A_{\nu}^{\rho} \left( \partial_{\rho} g_{\nu}^{\mu} / \partial_{\nu} \right), \quad (318)
\]

\[
\left( A_{\nu;\mu}^{\rho} \right)_{\nu}^{\mu} = \partial_{\nu} A_{\rho}^{\mu} + \partial_{\mu} A_{\nu}^{\rho} \left( \partial_{\rho} g_{\nu}^{\mu} / \partial_{\nu} \right), \quad (319)
\]

\[
\left( A_{\nu;\mu}^{\rho} \right)_{\nu}^{\mu} = \partial_{\nu} A_{\rho}^{\mu} + \partial_{\mu} A_{\nu}^{\rho} \left( \partial_{\rho} g_{\nu}^{\mu} / \partial_{\nu} \right), \quad (320)
\]
\[
(A^\nu_{\mu})_{\rho\sigma} = \frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} + \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\sigma} \right) ... (321)
\]

There isn’t index \( \nu \) in the second term of the right side of (318), (319), (320), (321) at all here. Therefore, I rewrite dummy index \( \sigma \) of (318), (319), (320), (321) in dummy index \( \nu \) and can get

\[
(A^\nu_{\mu})_{\rho\sigma} = \frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} + \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\nu} \right) ... (322)
\]

\[
(A^\nu_{\mu})_{\rho\sigma} = \frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} + \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\nu} \right) ... (323)
\]

\[
(A^\nu_{\mu})_{\rho\sigma} = \frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} + \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\nu} \right) ... (324)
\]

\[
(A^\nu_{\mu})_{\rho\sigma} = \frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} + \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\nu} \right) ... (325)
\]

By the way, by establishment of \( A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} \), (306), (314), (322) must be equal each other. (306), (314), (322) are equal each other in consideration of Definition 5. Furthermore, by establishment of \( A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} \), (307), (315), (323) aren’t equal each other in consideration of Definition 5. Thus, I decide not to handle (307), (315), (323). Furthermore, by establishment of \( A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} \), (308), (316), (324) must be equal each other. (308), (316), (324) aren’t equal each other in consideration of Definition 5. Thus, I decide not to handle (308), (316), (324). Furthermore, by establishment of \( A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} = A^\mu_{\nu,\rho,\sigma} \), (309), (317), (325) must be equal each other. (309), (317), (325) aren’t equal each other in consideration of Definition 5. Thus, I decide not to handle (309), (317), (325). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (114), (115), (116), (117) by Definition 3, Definision 2 and get

\[
-(A_{\mu}^\nu)^{\rho\sigma} = -\frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} - \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\sigma} \right) ... (326)
\]

\[
-(A_{\mu}^\nu)^{\rho\sigma} = -\frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} - \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\sigma} \right) ... (327)
\]

\[
-(A_{\mu}^\nu)^{\rho\sigma} = -\frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} - \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\sigma} \right) ... (328)
\]

\[
-(A_{\mu}^\nu)^{\rho\sigma} = -\frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} - \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\sigma} \right) ... (329)
\]

Because two same contravariant index existed in \( -\left( A^\nu_{\mu} \right)^{\rho\sigma} \) of (326), (327), (328), (329), I decided not to handle (326), (327), (328), (329). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (114), (115), (116), (117) by Definition 3, Definision 2 and get

\[
-(A^{\mu,\nu}_{\rho\sigma})_{\lambda\sigma} = -\frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} - \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\sigma} \right) ... (330)
\]

\[
-(A^{\mu,\nu}_{\rho\sigma})_{\lambda\sigma} = -\frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\sigma \partial x^\mu} - \frac{\partial^2 A^\nu}{\partial x^\rho \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\rho_\mu}{\partial x^\sigma} \right) ... (331)
\]
Because two same contravariant index existed in \(-\left( A^\mu;\nu;\mu \right)_{;\nu} \) of (330), (331), (332), (333), I decided not to handle (330), (331), (332), (333). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (114), (115), (116), (117) by Definition 3, Definition 2 and get

\[
\begin{align*}
-\left( A^\mu;\nu;\mu \right)_{;\nu} &= - \frac{\partial^3 A^\sigma}{\partial x_\mu \partial x_\mu \partial x_\mu} - \frac{\partial^2 A^\sigma}{\partial x_\mu \partial x_\mu} \left( \frac{\partial^2 g^\mu}{\partial x^\mu} \right) 2 \frac{\partial g^\mu}{\partial x_\mu} \ldots, \\
-\left( A^\mu;\nu;\mu \right)_{;\mu} &= - \frac{\partial^3 A^\sigma}{\partial x_\mu \partial x_\mu \partial x_\mu} - \frac{\partial^2 A^\sigma}{\partial x_\mu \partial x_\mu} \left( \frac{\partial^2 g^\mu}{\partial x^\mu} \right) 2 \frac{\partial g^\mu}{\partial x_\mu} \ldots.
\end{align*}
\]

Because two same contravariant index existed in \(-\left( A^\mu;\nu;\mu \right)_{;\mu} \) of (334), (335), (336), (337), I decided not to handle (334), (335), (336), (337). Similarly, I rewrite index \( \nu \) of the coordinate systems of (118) ~ (127) by Definition 3 and get

\[
\begin{align*}
-\left( A^\mu;\nu;\mu \right)_{;\nu} &= - \frac{\partial^3 A^\sigma}{\partial x_\mu \partial x_\mu \partial x_\mu} - \frac{\partial^2 A^\sigma}{\partial x_\mu \partial x_\mu} \left( \frac{\partial^2 g^\mu}{\partial x^\mu} \right) 2 \frac{\partial g^\mu}{\partial x_\mu} \ldots, \\
-\left( A^\mu;\nu;\mu \right)_{;\mu} &= - \frac{\partial^3 A^\sigma}{\partial x_\mu \partial x_\mu \partial x_\mu} - \frac{\partial^2 A^\sigma}{\partial x_\mu \partial x_\mu} \left( \frac{\partial^2 g^\mu}{\partial x^\mu} \right) 2 \frac{\partial g^\mu}{\partial x_\mu} \ldots.
\end{align*}
\]
\[-A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{345}\]

\[-A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{346}\]

\[-A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots. \tag{347}\]

Because two same covariant index existed in \(-A^\mu_{\mu;\mu;\mu}\) of (338) ~ (347), I decided not to handle (338) ~ (347). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (118) ~ (127) by Definision 2 and get

\[A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{348}\]

\[A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{349}\]

\[A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{350}\]

\[A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{351}\]

\[A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{352}\]

\[A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{353}\]

\[A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{354}\]

\[A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{355}\]

\[A^\mu_{\mu;\mu;\mu} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots. \tag{356}\]

Because two same contravariant index existed in \(A^\mu_{\mu;\mu;\mu}\) of (348) ~ (357), I decided not to handle (348) ~ (357). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (118) ~ (127) by Definision 3, Definision 2 and get

\[\left( A^\mu_{\mu;\mu;\mu} \right)^\mu = \frac{\partial^3 A^\nu}{\partial x^\rho \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\rho_\sigma}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right) \cdots, \tag{358}\]
There isn’t index \( \nu \) in the second term of the right side of (358) ~ (367) at all here. Therefore, I rewrite dummy index \( \sigma \) of (358) – (367) in dummy index \( \nu \) and can get

\[
\left( A_{\mu, \nu} \right)^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x_\mu} + \frac{\partial}{\partial x_\mu} \left( A^\nu \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \right) \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \ldots, (368)
\]

\[
\left( A_{\mu, \nu} \right)^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x_\mu} + \frac{\partial}{\partial x_\mu} \left( A^\nu \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \right) \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \ldots, (369)
\]

\[
\left( A_{\mu, \nu} \right)^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x_\mu} + \frac{\partial}{\partial x_\mu} \left( A^\nu \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \right) \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \ldots, (370)
\]

\[
\left( A_{\mu, \nu} \right)^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x_\mu} + \frac{\partial}{\partial x_\mu} \left( A^\nu \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \right) \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \ldots, (371)
\]

\[
\left( A_{\mu, \nu} \right)^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x_\mu} + \frac{\partial}{\partial x_\mu} \left( A^\nu \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \right) \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \ldots, (372)
\]

\[
\left( A_{\mu, \nu} \right)^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x_\mu} + \frac{\partial}{\partial x_\mu} \left( A^\nu \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \right) \frac{1}{2 \left( \frac{\partial g_{\nu \mu}}{\partial x^\mu} \right)} \ldots, (373)
\]
\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\nu}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\nu}} \right) \right], \quad (374)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\nu}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\nu}} \right) \right], \quad (375)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\nu}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\nu}} \right) \right], \quad (376)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\nu}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\nu}} \right) \right], \quad (377)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (118) \( \sim \) (127) by Definision 3, Definision 2 and get

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (378)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (379)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (380)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (381)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (382)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (383)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (384)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (385)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (386)
\]

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\sigma}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\sigma}} \right) \right], \quad (387)
\]

There isn’t index \( \nu \) in the second term of the right side of (378) \( \sim \) (387) at all here. Therefore, I rewrite dummy index \( \sigma \) of (378) \( \sim \) (387) in dummy index \( \nu \) and can get

\[
\left( A_{\mu,\nu}^{\mu} \right)^{\mu} = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} + \frac{\partial}{\partial x^{\nu}} \left[ A^{\mu} \left( \frac{1}{2} \left( \frac{\partial g_{\mu}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\nu}} \right) \right], \quad (388)
\]
\[
\begin{align*}
(A^\mu_{\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right), \quad (389) \\
(A^\mu_{\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right), \quad (390) \\
(A^\mu_{\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right), \quad (391) \\
(A^\mu_{\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right), \quad (392) \\
(A^\mu_{\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right), \quad (393) \\
(A^\mu_{\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right), \quad (394) \\
(A^\mu_{\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right), \quad (395) \\
(A^\mu_{\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right), \quad (396) \\
(A^\mu_{\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\sigma \partial x^\mu} + \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right), \quad (397)
\end{align*}
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (118) \(~ (127)\) by Definision 3, Definision 2 and get
\[
\begin{align*}
(A^{\mu\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right), \quad (398) \\
(A^{\mu\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right), \quad (399) \\
(A^{\mu\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right), \quad (400) \\
(A^{\mu\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right), \quad (401) \\
(A^{\mu\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right), \quad (402) \\
(A^{\mu\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right), \quad (403) \\
(A^{\mu\nu})^{(\mu}_{\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right), \quad (404)
\end{align*}
\]
\[
\left( A^{\mu}_{\nu;\rho}\right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\rho} + \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu \rho}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu \rho}}{\partial x^\mu} \right), \quad (405)
\]

\[
\left( A^{\mu}_{\nu;\rho}\right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\rho} + \frac{\partial}{\partial x^\rho} \left( A^\rho \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu \nu}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\rho \nu}}{\partial x^\mu} \right), \quad (406)
\]

\[
\left( A^{\mu}_{\nu;\rho}\right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\rho} + \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu \rho}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu \rho}}{\partial x^\mu} \right), \quad (407)
\]

There isn’t index \( \nu \) in the second term of the right side of (398) \( \sim \) (407) at all here. Therefore, I rewrite dummy index \( \sigma \) of (398) \( \sim \) (407) in dummy index \( \nu \) and can get

\[
\left( A^{\mu}_{\nu;\rho}\right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\rho} + \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu \rho}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu \rho}}{\partial x^\mu} \right), \quad (408)
\]

\[
\left( A^{\mu}_{\nu;\rho}\right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\rho} + \frac{\partial}{\partial x^\rho} \left( A^\rho \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu \nu}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\rho \nu}}{\partial x^\mu} \right), \quad (409)
\]

\[
\left( A^{\mu}_{\nu;\rho}\right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\rho} + \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu \rho}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu \rho}}{\partial x^\mu} \right), \quad (410)
\]

\[
\left( A^{\mu}_{\nu;\rho}\right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\nu \partial x^\rho} + \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu \rho}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu \rho}}{\partial x^\mu} \right), \quad (411)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (118) \( \sim \) (127) by Definition 3, Definition 2 and get

\[
-\left( A^{\mu}_{\nu;\rho}\right) = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho \partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu \rho}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu \rho}}{\partial x^\mu} \right), \quad (418)
\]

\[
-\left( A^{\mu}_{\nu;\rho}\right) = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\rho \partial x^\mu} - \frac{\partial}{\partial x^\rho} \left( A^\rho \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu \nu}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\rho \nu}}{\partial x^\mu} \right), \quad (419)
\]
Because two same contravariant index existed in $A_{\mu \rho}^{\nu \mu}$ of (418) ~ (427), I decided not to handle (418) ~ (427). Furthermore, I rewrite index $\nu$ of the coordinate systems of (118) ~ (127) by Definition 3, Definition 2 and get

$$\frac{\partial A_{\mu \rho}^{\nu \mu}}{\partial x_{\sigma}} = \frac{\partial A_{\mu \rho}^{\nu \mu}}{\partial x_{\sigma}} = \frac{\partial A_{\mu \rho}^{\nu \mu}}{\partial x_{\sigma}} = \frac{\partial A_{\mu \rho}^{\nu \mu}}{\partial x_{\sigma}} = \frac{\partial A_{\mu \rho}^{\nu \mu}}{\partial x_{\sigma}} = \frac{\partial A_{\mu \rho}^{\nu \mu}}{\partial x_{\sigma}} = \frac{\partial A_{\mu \rho}^{\nu \mu}}{\partial x_{\sigma}} = \frac{\partial A_{\mu \rho}^{\nu \mu}}{\partial x_{\sigma}} = \frac{\partial A_{\mu \rho}^{\nu \mu}}{\partial x_{\sigma}}$$

(428)
Because two same contravariant index existed in \( -\left( A^{\mu,\nu,\mu}_\mu \right)_\mu \) of (428) \( \sim \) (437), I decided not to handle (428) \( \sim \) (437). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (118) \( \sim \) (127) by Definition 3, Definition 2 and get

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (436)
\]

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (437)
\]

Because two same contravariant index existed in \( -\left( A^{\mu,\nu,\mu}_\mu \right)_\mu \) of (428) \( \sim \) (437), I decided not to handle (428) \( \sim \) (437). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (118) \( \sim \) (127) by Definition 3, Definition 2 and get

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (438)
\]

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (439)
\]

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (440)
\]

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (441)
\]

Because two same contravariant index existed in \( -\left( A^{\mu,\nu,\mu}_\mu \right)_\mu \) of (428) \( \sim \) (437), I decided not to handle (428) \( \sim \) (437). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (118) \( \sim \) (127) by Definition 3, Definition 2 and get

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (442)
\]

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (443)
\]

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (444)
\]

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (445)
\]

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (446)
\]

\[
-\left( A^{\mu,\nu,\mu}_\mu \right)_\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\sigma_{\mu,\mu}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\nu,\nu}}{\partial x^\mu} \right), \quad (447)
\]
Because two same covariant index existed in $-A_{\mu;\nu;\rho}^{\mu}$ of (448) ~ (457), I decided not to handle (448) ~ (457). Furthermore, I rewrite index $\nu$ of the coordinate systems of (128) ~ (137) by Definition 2 and get

$$A_{\mu;\nu;\rho}^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^{\sigma}}{\partial x_\mu} \left( \frac{\partial g^{\mu}_{\sigma}}{\partial x_\mu} \right) \left( \frac{\partial g^{\mu}_{\sigma}}{\partial x_\mu} \right) - \frac{\partial^3 A^{\mu}}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^{\sigma}}{\partial x_\mu} \left( \frac{\partial g^{\mu}_{\sigma}}{\partial x_\mu} \right) \left( \frac{\partial g^{\mu}_{\sigma}}{\partial x_\mu} \right),$$

(458)
Because two same contravariant index existed in $A_{\mu;\mu;\mu;\mu}^{\nu}$ of (458) $\sim$ (467), I decided not to handle (458) $\sim$ (467).

Furthermore, I rewrite index $\nu$ of the coordinate systems of (128) $\sim$ (137) by Definition 3, Definition 2 and get

\[
(A_{\mu;\mu;\mu;\mu})^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\nu}}{\partial x^{\sigma}} \ldots \tag{468}
\]

\[
(A_{\mu;\mu;\mu;\mu})^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\mu}}{\partial x^{\sigma}} \ldots \tag{469}
\]

\[
(A_{\mu;\mu;\mu;\mu})^{\ldots} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\ldots}}{\partial x^{\sigma}} \ldots \tag{470}
\]

\[
(A_{\mu;\mu;\mu;\mu})^{\ldots} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\ldots}}{\partial x^{\sigma}} \ldots \tag{471}
\]

\[
(A_{\mu;\mu;\mu;\mu})^{\ldots} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\ldots}}{\partial x^{\sigma}} \ldots \tag{472}
\]

\[
(A_{\mu;\mu;\mu;\mu})^{\ldots} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\ldots}}{\partial x^{\sigma}} \ldots \tag{473}
\]

\[
(A_{\mu;\mu;\mu;\mu})^{\ldots} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\ldots}}{\partial x^{\sigma}} \ldots \tag{474}
\]

There isn’t index $\nu$ in the second term of the right side of (468) $\sim$ (477) at all here. Therefore, I rewrite dummy index $\sigma$ of (468) $\sim$ (477) in dummy index $\nu$ and can get

\[
(A_{\mu;\mu;\mu;\mu})^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\nu}}{\partial x^{\sigma}} \ldots \tag{478}
\]

\[
(A_{\mu;\mu;\mu;\mu})^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\nu}}{\partial x^{\sigma}} \ldots \tag{479}
\]

\[
(A_{\mu;\mu;\mu;\mu})^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial A^{\sigma}}{\partial x_{\mu}} \left( \frac{\hat{g}_{\mu}^{\sigma}}{2} \right) \frac{\partial g_{\mu}^{\nu}}{\partial x^{\sigma}} \ldots \tag{480}
\]
Furthermore, I rewrite index \( \nu \) of the coordinate systems of (128) \(-\) (137) by Definition 3, Definition 2 and get

\[
\begin{align*}
\left( A_{\mu, \nu}^{\sigma} \right)^{\mu} & = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\nu} \partial x_{\mu}^{\nu}} - 2 \frac{\partial A^{\sigma}}{\partial x^{\nu}} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x_{\nu}^{\mu}} \right) \left( \frac{\partial g_{\nu}^{\mu}}{\partial x^{\nu}} \right), \\
\left( A_{\mu, \nu}^{\sigma} \right)^{\mu} & = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\nu} \partial x_{\mu}^{\nu}} - 2 \frac{\partial A^{\sigma}}{\partial x^{\nu}} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x_{\nu}^{\mu}} \right) \left( \frac{\partial g_{\nu}^{\mu}}{\partial x^{\nu}} \right), \\
\left( A_{\mu, \nu}^{\sigma} \right)^{\mu} & = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\nu} \partial x_{\mu}^{\nu}} - 2 \frac{\partial A^{\sigma}}{\partial x^{\nu}} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x_{\nu}^{\mu}} \right) \left( \frac{\partial g_{\nu}^{\mu}}{\partial x^{\nu}} \right), \\
\left( A_{\mu, \nu}^{\sigma} \right)^{\mu} & = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\nu} \partial x_{\mu}^{\nu}} - 2 \frac{\partial A^{\sigma}}{\partial x^{\nu}} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x_{\nu}^{\mu}} \right) \left( \frac{\partial g_{\nu}^{\mu}}{\partial x^{\nu}} \right), \\
\left( A_{\mu, \nu}^{\sigma} \right)^{\mu} & = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\nu} \partial x_{\mu}^{\nu}} - 2 \frac{\partial A^{\sigma}}{\partial x^{\nu}} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x_{\nu}^{\mu}} \right) \left( \frac{\partial g_{\nu}^{\mu}}{\partial x^{\nu}} \right), \\
\left( A_{\mu, \nu}^{\sigma} \right)^{\mu} & = \frac{\partial^{3} A^{\mu}}{\partial x^{\mu} \partial x^{\nu} \partial x_{\mu}^{\nu}} - 2 \frac{\partial A^{\sigma}}{\partial x^{\nu}} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x_{\nu}^{\mu}} \right) \left( \frac{\partial g_{\nu}^{\mu}}{\partial x^{\nu}} \right).
\end{align*}
\]
\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (497)
\]

There isn't index \( \nu \) in the second term of the right side of (488) \sim (497) at all here. Therefore, I rewrite dummy index \( \sigma \) of (488) \sim (497) in dummy index \( \nu \) and can get

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (498)
\]

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (499)
\]

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (500)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (128) \sim (137) by Definition 3, Definition 2 and get

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (501)
\]

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (502)
\]

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (503)
\]

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (504)
\]

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (505)
\]

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (506)
\]

\[
\left( A^\mu \right)_\mu^\nu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\sigma} \left( \frac{\partial g^\sigma}{\partial x^\mu} \right) \frac{1}{2} \left( \frac{\partial g^\mu}{\partial x^\mu} \right) \ldots, \quad (507)
\]
\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (512)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (513)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\nu}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (514)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (515)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\nu}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (516)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (517)
\]

There isn’t index \( \nu \) in the second term of the right side of (508) ~ (517) at all here. Therefore, I rewrite dummy index \( \sigma \) of (508) ~ (517) in dummy index \( \nu \) and can get

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\nu}{\partial x^\nu} \right), \quad (518)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (519)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\nu}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (520)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\nu}{\partial x^\nu} \right), \quad (521)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (522)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (523)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\nu}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (524)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (525)
\]

\[
\left( A^{\mu,\nu}_{\mu,\mu} \right) = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial A^\mu}{\partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\nu}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (526)
\]
Furthermore, I rewrite index $\nu$ of the coordinate systems of (128) – (137) by Definition 3, Definition 2 and get

\begin{align*}
-(A_{\mu}^{\mu,\nu})_{\mu,\nu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\mu} \left( \frac{1}{2} \frac{\partial g_{\sigma \mu}}{\partial x^\nu} \right) \left( \frac{1}{2} \frac{\partial g_{\mu \nu}}{\partial x^\mu} \right), \\
-(A_{\mu}^{\nu,\mu})_{\mu,\nu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{1}{2} \frac{\partial g_{\sigma \nu}}{\partial x^\mu} \right) \left( \frac{1}{2} \frac{\partial g_{\mu \nu}}{\partial x^\nu} \right), \\
-(A_{\mu}^{\nu,\nu})_{\mu,\nu} &= -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu} + 2 \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{1}{2} \frac{\partial g_{\sigma \nu}}{\partial x^\nu} \right) \left( \frac{1}{2} \frac{\partial g_{\nu \nu}}{\partial x^\nu} \right), \\
-(A_{\mu}^{\nu,\mu})_{\mu,\nu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{1}{2} \frac{\partial g_{\sigma \mu}}{\partial x^\nu} \right) \left( \frac{1}{2} \frac{\partial g_{\mu \mu}}{\partial x^\mu} \right),
\end{align*}

Because two same contravariant index existed in $-(A_{\mu}^{\mu,\nu})_{\mu,\nu}$ of (528) ~ (537), I decided not to handle (528) ~ (537). Furthermore, I rewrite index $\nu$ of the coordinate systems of (128) ~ (137) by Definition 3, Definition 2 and get

\begin{align*}
-(A_{\mu}^{\mu,\nu})_{\nu,\mu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{1}{2} \frac{\partial g_{\sigma \mu}}{\partial x^\nu} \right) \left( \frac{1}{2} \frac{\partial g_{\mu \nu}}{\partial x^\mu} \right), \\
-(A_{\mu}^{\nu,\mu})_{\nu,\mu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{1}{2} \frac{\partial g_{\sigma \nu}}{\partial x^\mu} \right) \left( \frac{1}{2} \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right), \\
-(A_{\mu}^{\nu,\nu})_{\nu,\mu} &= -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu} + 2 \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{1}{2} \frac{\partial g_{\sigma \nu}}{\partial x^\nu} \right) \left( \frac{1}{2} \frac{\partial g_{\nu \nu}}{\partial x^\nu} \right), \\
-(A_{\mu}^{\nu,\mu})_{\nu,\nu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\nu} \left( \frac{1}{2} \frac{\partial g_{\sigma \mu}}{\partial x^\nu} \right) \left( \frac{1}{2} \frac{\partial g_{\mu \nu}}{\partial x^\mu} \right).
\end{align*}
Because two same contravariant index existed in \(-\left( A^{\mu,\nu,\mu}_{\cdot\cdot}\right)_{\mu}^{\cdot}\) of (538) \sim (547), I decided not to handle (538) \sim (547). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (128) \sim (137) by Definition 3, Definition 2 and get

\[-\left( A^{\mu,\nu,\mu}_{\cdot\cdot}\right)_{\cdot\cdot}\mu = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\mu} \left( \frac{\partial g^{\sigma\mu}}{\partial x^\mu} \right) \left( \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right),\quad (542)\]

\[-\left( A^{\mu,\nu,\mu}_{\cdot\cdot}\right)_{\cdot\cdot}^{\cdot}\mu = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\mu} \left( \frac{\partial g^{\sigma\mu}}{\partial x^\mu} \right) \left( \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right),\quad (543)\]

\[-\left( A^{\mu,\nu,\mu}_{\cdot\cdot}\right)_{\cdot\cdot}\cdot\mu = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\mu} \left( \frac{\partial g^{\sigma\mu}}{\partial x^\mu} \right) \left( \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right),\quad (544)\]

\[-\left( A^{\mu,\nu,\mu}_{\cdot\cdot}\right)_{\cdot\cdot}^{\cdot}^{\cdot}\mu = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\mu} \left( \frac{\partial g^{\sigma\mu}}{\partial x^\mu} \right) \left( \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right),\quad (545)\]

\[-\left( A^{\mu,\nu,\mu}_{\cdot\cdot}\right)_{\cdot\cdot}^{\cdot}^{\cdot}\cdot\mu = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\mu} \left( \frac{\partial g^{\sigma\mu}}{\partial x^\mu} \right) \left( \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right),\quad (546)\]

\[-\left( A^{\mu,\nu,\mu}_{\cdot\cdot}\right)_{\cdot\cdot}^{\cdot}^{\cdot}^{\cdot}\mu = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial A^\sigma}{\partial x^\mu} \left( \frac{\partial g^{\sigma\mu}}{\partial x^\mu} \right) \left( \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right),\quad (547)\]
Because two same contravariant index existed in \( -A_{\nu,\mu;\mu}^{\mu} \) of (548) ~ (557), I decided not to handle (548) ~ (557). Similarly, I rewrite index \( \nu \) of the coordinate systems of (138), (139), (140), (141) by Definition 3 and get

\[\begin{align*}
-A_{\mu,\nu;\mu}^{\mu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\sigma_\mu}{\partial x^\mu} \right), \quad (558) \\
-A_{\mu,\nu;\mu}^{\mu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\sigma_\mu}{\partial x^\sigma} \right), \quad (559) \\
-A_{\mu,\nu;\mu}^{\mu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (560) \\
-A_{\mu,\nu;\mu}^{\mu} &= -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial^2 A^\mu}{\partial x^\mu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x^\mu} \right), \quad (561)
\end{align*}\]

Because two same covariant index existed in \( A_{\mu,\nu;\mu}^{\mu} \) of (558), (559), (560), (561), I decided not to handle (558), (559), (560), (561). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (138), (139), (140), (141) by Definition 2 and get

\[\begin{align*}
A_{\mu,\nu;\mu}^{\mu} &= \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial^2 A^\mu}{\partial x_\mu \partial x_\mu} \frac{1}{2} \left( \frac{\partial g^\gamma_\mu}{\partial x_\mu} \right), \quad (562) \\
A_{\mu,\nu;\mu}^{\mu} &= \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial^2 A^\mu}{\partial x_\mu \partial x_\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\gamma}{\partial x_\mu} \right), \quad (563) \\
A_{\mu,\nu;\mu}^{\mu} &= \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial^2 A^\mu}{\partial x_\mu \partial x_\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x_\mu} \right), \quad (564) \\
A_{\mu,\nu;\mu}^{\mu} &= \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial^2 A^\mu}{\partial x_\mu \partial x_\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x_\mu} \right), \quad (565)
\end{align*}\]

Because two same contravariant index existed in \( A_{\mu,\nu;\mu}^{\mu} \) of (562), (563), (564), (565), I decided not to handle (562), (563), (564), (565). Furthermore, I rewrite index \( \nu \) of the coordinate systems of (138), (139), (140), (141) by Definition 3, Definition 2 and get

\[\begin{align*}
\left(A_{\mu,\nu;\mu}^{\mu}\right)^{\mu} &= \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial^2 A^\mu}{\partial x_\mu \partial x_\mu} \frac{1}{2} \left( \frac{\partial g^\gamma_\mu}{\partial x_\mu} \right), \quad (566) \\
\left(A_{\mu,\nu;\mu}^{\mu}\right)^{\mu} &= \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial^2 A^\mu}{\partial x_\mu \partial x_\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\gamma}{\partial x_\mu} \right), \quad (567) \\
\left(A_{\mu,\nu;\mu}^{\mu}\right)^{\mu} &= \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial^2 A^\mu}{\partial x_\mu \partial x_\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x_\mu} \right), \quad (568) \\
\left(A_{\mu,\nu;\mu}^{\mu}\right)^{\mu} &= \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial^2 A^\mu}{\partial x_\mu \partial x_\mu} \frac{1}{2} \left( \frac{\partial g^\mu_\mu}{\partial x_\mu} \right), \quad (569)
\end{align*}\]

There isn’t index \( \nu \) in the second term of the right side of (566), (567), (568), (569) at all here. Therefore, I rewrite dummy index \( \sigma \) of (566), (567), (568), (569) in dummy index \( \nu \) and can get
\[
\left( A_{\mu;\mu}^{\nu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (570)
\]

\[
\left( A_{\mu;\mu}^{\nu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (571)
\]

\[
\left( A_{\mu;\mu}^{\nu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x^{\mu} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (572)
\]

\[
\left( A_{\mu;\mu}^{\nu} \right)^{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x^{\mu} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (573)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (138), (139), (140), (141) by Definition 3, Definitions 2 and get

\[
\left( A_{\nu}^{\mu} \right)_{\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\mu}} \right), \quad (574)
\]

\[
\left( A_{\nu}^{\mu} \right)_{\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x_{\mu} \partial x^{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\mu}} \right), \quad (575)
\]

\[
\left( A_{\nu}^{\mu} \right)_{\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x_{\mu} \partial x^{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x^{\mu} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (576)
\]

\[
\left( A_{\nu}^{\mu} \right)_{\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x_{\mu} \partial x^{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x^{\mu} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (577)
\]

There isn’t index \( \nu \) in the second term of the right side of (574), (575), (576), (577) at all here. Therefore, I rewrite dummy index \( \sigma \) of (574), (575), (576), (577) in dummy index \( \nu \) and can get

\[
\left( A_{\nu}^{\mu} \right)_{\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x_{\mu} \partial x^{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\mu}} \right), \quad (578)
\]

\[
\left( A_{\nu}^{\mu} \right)_{\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x_{\mu} \partial x^{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (579)
\]

\[
\left( A_{\nu}^{\mu} \right)_{\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x_{\mu} \partial x^{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (580)
\]

\[
\left( A_{\nu}^{\mu} \right)_{\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x_{\mu} \partial x^{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x^{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (581)
\]

Furthermore, I rewrite index \( \nu \) of the coordinate systems of (138), (139), (140), (141) by Definition 3, Definition 2 and get

\[
\left( A^{\mu;\mu} \right)_{\mu;\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x^{\mu} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x^{\mu}} \right), \quad (582)
\]

\[
\left( A^{\mu;\mu} \right)_{\mu;\mu}^{\nu} = \frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x^{\mu} \partial x_{\mu}} - 2 \frac{\partial^2 A^{\mu}}{\partial x^{\mu} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\nu}}{\partial x^{\mu}} \right), \quad (583)
\]
(A^{\mu;\nu};_{\mu;\nu}) = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} - 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\sigma \mu}}{\partial x^\sigma} \right), \quad (584)

(A^{\mu;\nu};_{\mu;\nu}) = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} - 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\sigma \mu}}{\partial x^\sigma} \right), \quad (585)

There isn’t index \( \nu \) in the second term of the right side of (582), (583), (584), (585) at all here. Therefore, I rewrite dummy index \( \sigma \) of (582), (583), (584), (585) in dummy index \( \nu \) and can get

\[
(A^{\mu;\nu};_{\mu;\nu}) = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} - 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\nu \mu}}{\partial x^\nu} \right), \quad (586)
\]

\[
(A^{\mu;\nu};_{\mu;\nu}) = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} - 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\nu \mu}}{\partial x^\nu} \right), \quad (587)
\]

\[
(A^{\mu;\nu};_{\mu;\nu}) = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} - 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\nu \mu}}{\partial x^\nu} \right), \quad (588)
\]

\[
(A^{\mu;\nu};_{\mu;\nu}) = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} - 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\nu \mu}}{\partial x^\nu} \right), \quad (589)
\]

By the way, by establishment of \( A^{\mu}_{\nu;\rho;\sigma} = A^{\mu}_{\rho;\nu;\sigma} = A^{\mu}_{\sigma;\nu;\rho} = A^{\mu}_{\rho;\sigma;\nu} = A^{\mu}_{\sigma;\rho;\nu}, \quad (570), (578), (586) \) must be equal each other. \( (570), (578), (586) \) are equal each other in consideration of Definision 5. Furthermore, by establishment of \( A^{\mu}_{\nu;\sigma;\rho} = A^{\mu}_{\sigma;\nu;\rho} = A^{\mu}_{\rho;\sigma;\nu} = A^{\mu}_{\rho;\nu;\sigma} = A^{\mu}_{\sigma;\rho;\nu}, \quad (571), (579), (587) \) must be equal each other. \( (571), (579), (587) \) aren’t equal each other in consideration of Definision 5. Thus, I decide not to handle \( (571), (579), (587) \). Furthermore, by establishment of \( A^{\mu}_{\nu;\sigma;\rho} = A^{\mu}_{\rho;\nu;\sigma} = A^{\mu}_{\sigma;\rho;\nu} = A^{\mu}_{\rho;\sigma;\nu} = A^{\mu}_{\sigma;\rho;\nu}, \quad (572), (580), (588) \) must be equal each other. \( (572), (580), (588) \) aren’t equal each other in consideration of Definision 5. Thus, I decide not to handle \( (572), (580), (588) \). Furthermore, by establishment of \( A^{\mu}_{\nu;\sigma;\rho} = A^{\mu}_{\rho;\nu;\sigma} = A^{\mu}_{\sigma;\rho;\nu} = A^{\mu}_{\rho;\sigma;\nu} = A^{\mu}_{\sigma;\rho;\nu}, \quad (573), (581), (589) \) must be equal each other. \( (573), (581), (589) \) aren’t equal each other in consideration of Definision 5. Thus, I decide not to handle \( (573), (581), (589) \). Furthermore, I rewrite index \( \nu \) of the coordinate systems of \( (138), (139), (140), (141) \) by Definision 3, Definision 2 and get

\[
-(A^{\mu};_{\nu};\mu) = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} + 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\sigma \mu}}{\partial x^\sigma} \right), \quad (590)
\]

\[
-(A^{\mu};_{\nu};\mu) = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} + 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\sigma \mu}}{\partial x^\sigma} \right), \quad (591)
\]

\[
-(A^{\mu};_{\nu};\mu) = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} + 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\sigma \mu}}{\partial x^\sigma} \right), \quad (592)
\]

\[
-(A^{\mu};_{\nu};\mu) = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\nu \partial x^\mu} + 2 \frac{\partial^2 A^{\mu}}{\partial x^\nu \partial x^\mu} \frac{1}{2} \left( \frac{\partial g^{\sigma \mu}}{\partial x^\sigma} \right), \quad (593)
\]

Because two same contravariant index existed in \( -(A^{\mu};_{\nu};\mu) \) of \( (590), (591), (592), (593) \), I decided not to handle \( (590), (591), (592), (593) \). Furthermore, I rewrite index \( \nu \) of the coordinate systems of \( (138), (139), (140), (141) \) by Definision 3, Definision 2 and get
\[
-(A^{\mu,\nu,\mu})_{\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (594)
\]
\[
-(A^{\mu,\nu,\mu})_{\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (595)
\]
\[
-(A^{\mu,\nu,\mu})_{\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (596)
\]
\[
-(A^{\mu,\nu,\mu})_{\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (597)
\]

Because two same contravariant index existed in \((A^{\mu,\nu,\mu})_{\mu}\) of (594), (595), (596), (597), I decided not to handle (594), (595), (596), (597). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (138), (139), (140), (141) by Definition 3, Definition 2 and get

\[
-(A^{\mu,\nu,\mu})_{\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (598)
\]
\[
-(A^{\mu,\nu,\mu})_{\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (599)
\]
\[
-(A^{\mu,\nu,\mu})_{\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (600)
\]
\[
-(A^{\mu,\nu,\mu})_{\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\mu} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\mu} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (601)
\]

Because two same contravariant index existed in \((A^{\mu,\nu,\mu})_{\mu}\) of (598), (599), (600), (601), I decided not to handle (598), (599), (600), (601). Similarly, I rewrite index \(\nu\) of the coordinate systems of (142) ~ (151) by Definition 3 and get

\[
-A^{\mu}_{\sigma,\nu,\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\sigma} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\sigma} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (602)
\]
\[
-A^{\mu}_{\sigma,\nu,\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\sigma} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\sigma} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (603)
\]
\[
-A^{\mu}_{\sigma,\nu,\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\sigma} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\sigma} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (604)
\]
\[
-A^{\mu}_{\sigma,\nu,\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\sigma} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\sigma} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (605)
\]
\[
-A^{\mu}_{\sigma,\nu,\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\sigma} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\sigma} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (606)
\]
\[
-A^{\mu}_{\sigma,\nu,\mu} = -\frac{\partial^3 A^{\mu}}{\partial x_{\sigma} \partial x_{\mu} \partial x_{\mu}} + 2 \frac{\partial^2 A^{\mu}}{\partial x_{\sigma} \partial x_{\sigma} \partial x_{\mu}} \frac{1}{2} \left( \frac{\partial g_{\sigma}}{\partial x_{\mu}} \right), \quad (607)
\]
\[ -A_{\mu,\mu;\mu}^\alpha = -\frac{\partial^3 A^\alpha}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\sigma} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (608) \]

\[ -A_{\mu,\mu;\mu}^\mu = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\sigma} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (609) \]

\[ -A_{\mu,\mu;\mu}^\sigma = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\sigma} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (610) \]

\[ -A_{\mu,\mu;\mu}^\rho = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\sigma} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (611) \]

Because two same covariant index existed in \(-A_{\mu,\mu;\mu}^\mu\) of (602) ~ (611), I decided not to handle (602) ~ (611). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (142) ~ (151) by definition 2 and get

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = -\frac{\partial^3 A^\mu}{\partial x^{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (612) \]

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = \frac{\partial^3 A^\mu}{\partial x^{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (613) \]

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = -\frac{\partial^3 A^\mu}{\partial x^{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (614) \]

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = \frac{\partial^3 A^\mu}{\partial x^{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (615) \]

Because two same contravariant index existed in \(A_{\mu;\mu;\mu}^{\mu,\nu}\) of (612) ~ (621), I decided not to handle (612) ~ (621). Furthermore, I rewrite index \(\nu\) of the coordinate systems of (142) ~ (151) by definition 3, definition 2 and get

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = -\frac{\partial^3 A^\mu}{\partial x_{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (616) \]

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = \frac{\partial^3 A^\mu}{\partial x^{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (617) \]

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = -\frac{\partial^3 A^\mu}{\partial x^{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (618) \]

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = \frac{\partial^3 A^\mu}{\partial x^{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (619) \]

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = -\frac{\partial^3 A^\mu}{\partial x^{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (620) \]

\[ A_{\mu;\mu;\mu}^{\mu,\nu} = \frac{\partial^3 A^\mu}{\partial x^{\mu,\mu,\mu}} - 2 \frac{\partial}{\partial x^{\sigma}} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g_{\mu \sigma}}{\partial x^\sigma} \right), \quad (621) \]
\[
\begin{align*}
(A^{\mu}_{i,j,k})_{\nu} &= -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial}{\partial x^\mu} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\mu}_{\nu}}{\partial x^\mu} \right) \right) - \frac{1}{2} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x^\mu} \right), \quad (622) \\
(A^{\mu}_{i,j,k})_{\nu} &= -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial}{\partial x^\mu} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\mu}_{\nu}}{\partial x^\mu} \right) \right) - \frac{1}{2} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x^\mu} \right), \quad (623) \\
(A^{\mu}_{i,j,k})_{\nu} &= -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial}{\partial x^\mu} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\mu}_{\nu}}{\partial x^\mu} \right) \right) - \frac{1}{2} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x^\mu} \right), \quad (624) \\
(A^{\mu}_{i,j,k})_{\nu} &= -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial}{\partial x^\mu} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\mu}_{\nu}}{\partial x^\mu} \right) \right) - \frac{1}{2} \left( \frac{\partial g_{\mu}^{\sigma}}{\partial x^\mu} \right), \quad (625)
\end{align*}
\]

There isn’t index \( \nu \) in the second term of the right side of (622) \( \sim \) (631) at all here. Therefore, I rewrite dummy index \( \sigma \) of (622) \( \sim \) (631) in dummy index \( \nu \) and can get

\[
\begin{align*}
(A^{\mu}_{i,j,k})_{\nu} &= -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial}{\partial x^\mu} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\mu}_{\nu}}{\partial x^\mu} \right) \right) - \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x^\mu} \right), \quad (632) \\
(A^{\mu}_{i,j,k})_{\nu} &= -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial}{\partial x^\mu} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\mu}_{\nu}}{\partial x^\mu} \right) \right) - \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x^\mu} \right), \quad (633) \\
(A^{\mu}_{i,j,k})_{\nu} &= -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial}{\partial x^\mu} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\mu}_{\nu}}{\partial x^\mu} \right) \right) - \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x^\mu} \right), \quad (634) \\
(A^{\mu}_{i,j,k})_{\nu} &= -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial}{\partial x^\mu} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\mu}_{\nu}}{\partial x^\mu} \right) \right) - \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x^\mu} \right), \quad (635) \\
(A^{\mu}_{i,j,k})_{\nu} &= -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^\mu \partial x^\mu} - 2 \frac{\partial}{\partial x^\mu} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\mu}_{\nu}}{\partial x^\mu} \right) \right) - \frac{1}{2} \left( \frac{\partial g_{\mu}^{\nu}}{\partial x^\mu} \right), \quad (636)
\end{align*}
\]
Furthermore, I rewrite index $\nu$ of the coordinate systems of (142) – (151) by Definition 3, Definition 2 and get

\[
(A^\mu_{\nu\mu\nu})^\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu_{\nu\mu}}{\partial x^\nu} \right) \right) + \frac{1}{2} \left( \frac{\partial g^\nu_{\nu\nu}}{\partial x^\nu} \right), \quad (637)
\]

\[
(A^\mu_{\nu\mu\nu})^\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu_{\nu\mu}}{\partial x^\nu} \right) \right) + \frac{1}{2} \left( \frac{\partial g^\nu_{\nu\nu}}{\partial x^\nu} \right), \quad (638)
\]

\[
(A^\mu_{\nu\mu\nu})^\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu_{\nu\mu}}{\partial x^\nu} \right) \right) + \frac{1}{2} \left( \frac{\partial g^\nu_{\nu\nu}}{\partial x^\nu} \right), \quad (639)
\]

\[
(A^\mu_{\nu\mu\nu})^\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu_{\nu\mu}}{\partial x^\nu} \right) \right) + \frac{1}{2} \left( \frac{\partial g^\nu_{\nu\nu}}{\partial x^\nu} \right), \quad (640)
\]

\[
(A^\mu_{\nu\mu\nu})^\mu = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^\mu_{\nu\mu}}{\partial x^\nu} \right) \right) + \frac{1}{2} \left( \frac{\partial g^\nu_{\nu\nu}}{\partial x^\nu} \right), \quad (641)
\]
There isn’t index $\nu$ in the second term of the right side of (642) ~ (651) at all here. Therefore, I rewrite dummy index $\sigma$ of (642) ~ (651) in dummy index $v$ and can get

\[
(A^{\alpha}_{\mu})_{\mu} = \frac{\partial^3 A^{\alpha}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - 2 \frac{\partial}{\partial x^{\mu}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{v}_{\alpha}}{\partial x^{v}} \right) \ldots, \tag{652}
\]

\[
(A^{\mu}_{\alpha})_{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - 2 \frac{\partial}{\partial x^{v}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{v}_{\mu}}{\partial x^{\nu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{v}_{\mu}}{\partial x^{v}} \right) \ldots, \tag{653}
\]

\[
(A^{\mu}_{\alpha})_{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - 2 \frac{\partial}{\partial x^{v}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{v}_{\mu}}{\partial x^{\nu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{v}_{\mu}}{\partial x^{v}} \right) \ldots, \tag{654}
\]

\[
(A^{\mu}_{\alpha})_{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - 2 \frac{\partial}{\partial x^{v}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{v}_{\mu}}{\partial x^{\nu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{v}_{\mu}}{\partial x^{v}} \right) \ldots, \tag{655}
\]

\[
(A^{\mu}_{\alpha})_{\mu} = \frac{\partial^3 A^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} - 2 \frac{\partial}{\partial x^{v}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{v}_{\mu}}{\partial x^{\nu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{v}_{\mu}}{\partial x^{v}} \right) \ldots, \tag{656}
\]

Furthermore, I rewrite index $v$ of the coordinate systems of (142) ~ (151) by Definition 3, Definition 2 and get

\[
(A^{\alpha}_{\mu,\mu}) = \frac{\partial^3 A^{\alpha}}{\partial x^{\mu,\mu} \partial x^{\mu,\mu}} - 2 \frac{\partial}{\partial x^{\mu,\mu}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\mu,\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\nu}} \right) \ldots, \tag{662}
\]

\[
(A^{\mu}_{\alpha,\mu}) = \frac{\partial^3 A^{\mu}}{\partial x^{\mu,\mu} \partial x^{\mu,\mu}} - 2 \frac{\partial}{\partial x^{\mu,\mu}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\mu,\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\nu}} \right) \ldots, \tag{663}
\]

\[
(A^{\mu}_{\alpha,\mu}) = \frac{\partial^3 A^{\mu}}{\partial x^{\mu,\mu} \partial x^{\mu,\mu}} - 2 \frac{\partial}{\partial x^{\mu,\mu}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\mu,\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\nu}} \right) \ldots, \tag{664}
\]

\[
(A^{\mu}_{\alpha,\mu}) = \frac{\partial^3 A^{\mu}}{\partial x^{\mu,\mu} \partial x^{\mu,\mu}} - 2 \frac{\partial}{\partial x^{\mu,\mu}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\mu,\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\nu}} \right) \ldots, \tag{665}
\]

\[
(A^{\mu}_{\alpha,\mu}) = \frac{\partial^3 A^{\mu}}{\partial x^{\mu,\mu} \partial x^{\mu,\mu}} - 2 \frac{\partial}{\partial x^{\mu,\mu}} \left( A^{\nu} \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\mu,\mu}} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\alpha}}{\partial x^{\nu}} \right) \ldots, \tag{666}
\]
\begin{align}
(A^{\mu\nu})_{;\mu\mu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\nu} - 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^\nu_\sigma}{\partial x^\sigma} \right) \right), \tag{667} \\
(A^{\mu\nu})_{;\mu\sigma} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\sigma} - 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^\nu_\sigma}{\partial x^\sigma} \right) \right), \tag{668} \\
(A^{\mu\nu})_{;\nu\mu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^\nu_\sigma}{\partial x^\sigma} \right) \right), \tag{669} \\
(A^{\mu\nu})_{;\nu\sigma} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\sigma} - 2 \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^\nu_\sigma}{\partial x^\sigma} \right) \right), \tag{670} \\
(A^{\mu\nu})_{;\mu\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\nu} - 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_\sigma}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^\nu_\sigma}{\partial x^\sigma} \right) \right), \tag{671} \\
\end{align}

There isn’t index $v$ in the second term of the right side of (662) ~ (671) at all here. Therefore, I rewrite dummy index $\sigma$ of (662) ~ (671) in dummy index $v$ and can get

\begin{align}
(A^{\mu\nu})_{;\mu\mu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\nu} - 2 \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \right), \tag{672} \\
(A^{\mu\nu})_{;\mu\sigma} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\sigma} - 2 \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \right), \tag{673} \\
(A^{\mu\nu})_{;\nu\mu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \right), \tag{674} \\
(A^{\mu\nu})_{;\nu\sigma} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\sigma} - 2 \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \right), \tag{675} \\
(A^{\mu\nu})_{;\mu\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\nu} - 2 \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \right), \tag{676} \\
(A^{\mu\nu})_{;\nu\mu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \right), \tag{677} \\
(A^{\mu\nu})_{;\mu\nu} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\nu} - 2 \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \right), \tag{678} \\
(A^{\mu\nu})_{;\nu\nu} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} - 2 \frac{\partial}{\partial x^\nu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \right), \tag{679} \\
(A^{\mu\nu})_{;\nu\sigma} &= \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu \partial x^\sigma} - 2 \frac{\partial}{\partial x^\nu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \right), \tag{680} \\
(A^{\mu\nu})_{;\mu\sigma} &= \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\sigma} - 2 \frac{\partial}{\partial x^\mu} \left( A^\nu \frac{1}{2} \left( \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \right) \left( \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \right), \tag{681} \\
\end{align}
Furthermore, I rewrite index \( v \) of the coordinate systems of (142) \( \sim \) (151) by Definision 3, Definision 2 and get

\[
-(A_{\mu}^{\nu};_{\mu}) = -\frac{\partial^3 A^{\mu}}{\partial x^\nu \partial x^\sigma \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^{\sigma} \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\sigma} \right), \quad (682)
\]

\[
-(A_{\mu}^{\nu};_{\mu}) = -\frac{\partial^3 A^{\mu}}{\partial x^\nu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^{\sigma} \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\sigma} \right), \quad (683)
\]

\[
-(A_{\mu}^{\nu};_{\mu}) = -\frac{\partial^3 A^{\mu}}{\partial x^\nu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\sigma} \left( A^{\sigma} \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\sigma} \right), \quad (684)
\]

Because two same contravariant index existed in \( -(A_{\mu}^{\nu};_{\mu}) \) of (682) \( \sim \) (691), I decided not to handle (682) \( \sim \) (691). Furthermore, I rewrite index \( v \) of the coordinate systems of (142) \( \sim \) (151) by Definision 3, Definision 2 and get

\[
-(A_{\mu}^{\nu};_{\mu}) = \frac{\partial^3 A^{\mu}}{\partial x^\nu \partial x^\sigma \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^{\sigma} \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\sigma} \right), \quad (692)
\]

\[
-(A_{\mu}^{\nu};_{\mu}) = \frac{\partial^3 A^{\mu}}{\partial x^\nu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^{\sigma} \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\sigma} \right), \quad (693)
\]

\[
-(A_{\mu}^{\nu};_{\mu}) = \frac{\partial^3 A^{\mu}}{\partial x^\nu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\sigma} \left( A^{\sigma} \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\sigma} \right), \quad (694)
\]

\[
-(A_{\mu}^{\nu};_{\mu}) = \frac{\partial^3 A^{\mu}}{\partial x^\nu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\sigma} \left( A^{\sigma} \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^{\mu \sigma}}{\partial x^\sigma} \right), \quad (695)
\]
\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right), \quad \text{...} \] (696)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (697)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (698)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right), \quad \text{...} \] (699)

Because two same contravariant index existed in \(-(A^{\mu,\nu,\lambda})_{\mu}\) of (692)~(701), I decided not to handle (692)~(701). Furthermore, I rewrite index $v$ of the coordinate systems of (142)~(151) by Definision 3, Definision 2 and get

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (702)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (703)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (704)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (705)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (706)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (707)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (708)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\sigma} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (709)

\[-(A^{\mu,\nu,\lambda})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} + 2 \frac{\partial}{\partial x^\mu} \left( A^\sigma \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right) \right) \frac{1}{2} \left( \frac{\partial g^\mu_{\sigma}}{\partial x^\mu} \right), \quad \text{...} \] (710)
\[-(A^{\mu;\mu})_{\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} + 2 \frac{\partial}{\partial x^\sigma} \left( A^{\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (711)\]

Because two same contravariant index existed in \[-(A^{\mu;\mu})_{\mu;\mu}\] of (702) \sim (711), I decided not to handle (702) \sim (711).

Similarly, I rewrite index \(v\) of the coordinate systems of (152) \sim (161) by Definition 3 and get

\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (712)\]
\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (713)\]
\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (714)\]
\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (715)\]
\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (716)\]
\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (717)\]
\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (718)\]
\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (719)\]
\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (720)\]
\[-A^{\mu}_{\mu;\mu;\mu} = -\frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} - 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (721)\]

Because two same covariant index existed in \[-A^{\mu}_{\mu;\mu;\mu}\] of (712) \sim (721), I decided not to handle (712) \sim (721).

Furthermore, I rewrite index \(v\) of the coordinate systems of (152) \sim (161) by Definition 2 and get

\[A^{\mu;\mu;\mu;\mu} = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} + 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (722)\]
\[A^{\mu;\mu;\mu;\mu} = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} + 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (723)\]
\[A^{\mu;\mu;\mu;\mu} = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} + 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (724)\]
\[A^{\mu;\mu;\mu;\mu} = \frac{\partial^3 A^{\mu}}{\partial x^\mu \partial x^{\mu} \partial x^\mu} + 2 \frac{\partial A^{\mu}}{\partial x^\sigma} \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \frac{1}{2} \left( \hat{g}_{\sigma \sigma}^{\rho} \right) \ldots \quad (725)\]
Because two same contravariant index existed in $A_{\mu,\nu;\rho;\sigma}$ of (722) ~ (731), I decided not to handle (722) ~ (731). Furthermore, I rewrite index $\nu$ of the coordinate systems of (152) ~ (161) by Definition 3, Definition 2 and get

$$A_{\mu,\nu;\rho;\sigma} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\sigma \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g_\sigma^{\nu}}{\partial x_\rho} \right) \left( \frac{\partial g^{\nu\mu}}{\partial x_\sigma} \right), \quad (726)$$

$$A_{\mu,\nu;\rho;\sigma} = \frac{\partial^3 A^\mu}{\partial x_\rho \partial x^\nu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g_\sigma^{\nu}}{\partial x_\rho} \right) \left( \frac{\partial g^{\nu\mu}}{\partial x_\sigma} \right), \quad (727)$$

$$A_{\mu,\nu;\rho;\sigma} = \frac{\partial^3 A^\mu}{\partial x_\rho \partial x^\sigma \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g_\sigma^{\nu}}{\partial x_\rho} \right) \left( \frac{\partial g^{\nu\mu}}{\partial x_\sigma} \right), \quad (728)$$

$$A_{\mu,\nu;\rho;\sigma} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\sigma \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g_\sigma^{\nu}}{\partial x_\rho} \right) \left( \frac{\partial g^{\nu\mu}}{\partial x_\sigma} \right), \quad (729)$$

$$A_{\mu,\nu;\rho;\sigma} = \frac{\partial^3 A^\mu}{\partial x_\rho \partial x^\nu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g_\sigma^{\nu}}{\partial x_\rho} \right) \left( \frac{\partial g^{\nu\mu}}{\partial x_\sigma} \right), \quad (730)$$

$$A_{\mu,\nu;\rho;\sigma} = \frac{\partial^3 A^\mu}{\partial x^\rho \partial x^\sigma \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g_\sigma^{\nu}}{\partial x_\rho} \right) \left( \frac{\partial g^{\nu\mu}}{\partial x_\sigma} \right), \quad (731)$$
There isn’t index $\nu$ in the second term of the right side of (732) ~ (741) at all here. Therefore, I rewrite dummy index $\sigma$ of (732) ~ (741) in dummy index $\nu$ and can get

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (742)
\]

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (743)
\]

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (744)
\]

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (745)
\]

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (746)
\]

Furthermore, I rewrite index $\nu$ of the coordinate systems of (152) ~ (161) by Definition 3, Definition 2 and get

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (752)
\]

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (753)
\]

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (754)
\]

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (755)
\]

\[
(A_{\mu,\nu}^{\nu})^{\mu} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} + 2 \frac{\partial A^\mu}{\partial x^\mu} \left[ \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \frac{1}{2} \left( \frac{\partial g_{\nu \mu}}{\partial x^\nu} \right) \right], \quad (756)
\]
\begin{align}
\left( A'_{\mu} \right)_{\nu}^{\mu} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\mu} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\mu}}{\partial x^{\sigma}} \right) \frac{1}{2} \left( \frac{\partial g^{\sigma}_{\mu}}{\partial x^{\sigma}} \right), \quad (757) \\
\left( A''_{\mu} \right)_{\nu}^{\mu} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\mu} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\mu}}{\partial x^{\sigma}} \right) \frac{1}{2} \left( \frac{\partial g^{\sigma}_{\mu}}{\partial x^{\sigma}} \right), \quad (758) \\
\left( A'''_{\mu} \right)_{\nu}^{\mu} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\mu} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\mu}}{\partial x^{\sigma}} \right) \frac{1}{2} \left( \frac{\partial g^{\sigma}_{\mu}}{\partial x^{\sigma}} \right), \quad (759) \\
\left( A''''_{\mu} \right)_{\nu}^{\mu} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\mu} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\mu}}{\partial x^{\sigma}} \right) \frac{1}{2} \left( \frac{\partial g^{\sigma}_{\mu}}{\partial x^{\sigma}} \right), \quad (760) \\
\left( A''''''_{\mu} \right)_{\nu}^{\mu} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\mu} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\mu}}{\partial x^{\sigma}} \right) \frac{1}{2} \left( \frac{\partial g^{\sigma}_{\mu}}{\partial x^{\sigma}} \right). \quad (761)
\end{align}

There isn’t index \( \nu \) in the second term of the right side of (752) \sim (761) at all here. Therefore, I rewrite dummy index \( \sigma \) of (752) \sim (761) in dummy index \( \nu \) and can get

\begin{align}
\left( A'_{\sigma} \right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right), \quad (762) \\
\left( A''_{\sigma} \right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right), \quad (763) \\
\left( A'''_{\sigma} \right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right), \quad (764) \\
\left( A''''_{\sigma} \right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right), \quad (765) \\
\left( A''''''_{\sigma} \right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right), \quad (766) \\
\left( A''''''''_{\sigma} \right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right), \quad (767) \\
\left( A'''''''\right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right), \quad (768) \\
\left( A''''''''' \right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right), \quad (769) \\
\left( A'''''''''' \right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right), \quad (770) \\
\left( A''''''''''' \right)_{\nu}^{\sigma} &= \frac{\partial^3 A^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\mu}} + 2 \frac{\partial A^{\mu}}{\partial x^{\mu}} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right) \frac{1}{2} \left( \frac{\partial g^{\nu}_{\sigma}}{\partial x^{\nu}} \right). \quad (771)
\end{align}
Furthermore, I rewrite index $\nu$ of the coordinate systems of (152) ~ (161) by Definition 3, Definition 2 and get

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (772)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (773)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (774)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (775)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (776)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (777)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (778)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (779)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (780)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (781)
\]

There isn’t index $\nu$ in the second term of the right side of (772) ~ (781) at all here. Therefore, I rewrite dummy index $\sigma$ of (772) ~ (781) in dummy index $\nu$ and can get

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (782)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (783)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (784)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (785)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (786)
\]

\[
(A^{\mu,\nu})_{\mu,\mu} = \frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^{\mu \nu}}{\partial x_\mu} \right) \left( \frac{\partial g^{\nu \sigma}}{\partial x_\sigma} \right), \quad (787)
\]
Furthermore, I rewrite index $\nu$ of the coordinate systems of (152) $\sim$ (161) by Definision 3, Definision 2 and get

$$-(A^\mu)_{\nu;\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(792)

$$-(A^\mu)_{\nu;\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(793)

$$-(A^\mu)_{\nu;\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(794)

$$-(A^\mu)_{\nu;\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(795)

$$-(A^\mu)_{\nu;\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(796)

$$-(A^\mu)_{\nu;\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(797)

$$-(A^\mu)_{\nu;\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(798)

$$-(A^\mu)_{\nu;\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(799)

$$-(A^\mu)_{\nu;\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(800)

Because two same contravariant index existed in $-(A^\mu)_{\nu;\mu}$ of (792) $\sim$ (801), I decided not to handle (792) $\sim$ (801). Furthermore, I rewrite index $\nu$ of the coordinate systems of (152) $\sim$ (161) by Definision 3, Definision 2 and get

$$-(A^\mu;\nu)_{\mu} = -\frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\mu} - 2\frac{\partial A^\mu}{\partial x^\nu} \frac{1}{2} \left( \frac{\partial g^{\nu\sigma}}{\partial x^\sigma} \right) \frac{1}{2} \left( \frac{\partial g^{\mu\sigma}}{\partial x^\sigma} \right),$$

(802)
\[-(A_{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\rho} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(803)}\]

\[-(A_{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(804)}\]

\[-(A_{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(805)}\]

\[-(A_{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(806)}\]

\[-(A_{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(807)}\]

\[-(A_{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(808)}\]

\[-(A_{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(809)}\]

\[-(A_{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(810)}\]

\[-(A_{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(811)}\]

Because two same contravariant index existed in \(-(A^{\mu^{\epsilon},\mu^{\mu}}^{\rho})_{\mu}\) of (802) ~ (811), I decided not to handle (802) ~ (811). Furthermore, I rewrite index \(v\) of the coordinate systems of (152) ~ (161) by Definition 3, Definition 2 and get

\[-(A^{\mu^{\epsilon}}_{\mu^\mu})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(812)}\]

\[-(A^{\mu^{\epsilon}}_{\mu^\mu})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(813)}\]

\[-(A^{\mu^{\epsilon}}_{\mu^\mu})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(814)}\]

\[-(A^{\mu^{\epsilon}}_{\mu^\mu})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(815)}\]

\[-(A^{\mu^{\epsilon}}_{\mu^\mu})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(816)}\]

\[-(A^{\mu^{\epsilon}}_{\mu^\mu})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(817)}\]

\[-(A^{\mu^{\epsilon}}_{\mu^\mu})_{\mu} = -\frac{\partial^3 A^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - 2 \frac{\partial A^\mu}{\partial x_\mu} \left( \frac{\partial g^\sigma_{\mu \epsilon}}{\partial x_\mu} \right) \frac{1}{2} \left( \frac{\partial g^\epsilon_{\mu \sigma}}{\partial x_\mu} \right), \quad \text{(818)}\]
Because two same contravariant index existed in \((A^{\mu;\nu;\rho})_{\mu}^{\nu} \) of (812) ~ (821), I decided not to handle (812) ~ (821). And I get

\[
(A^{\mu;\nu;\rho})_{\mu}^{\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} \quad (822)
\]

from (174), (218), (262), (306), (368) ~ (377), (478) ~ (487), (570), (632) ~ (641), (742) ~ (751) in consideration of Definition 5. And I get

\[
(A^{\mu;\nu;\rho})_{\mu}^{\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} \quad (823)
\]

from (182), (226), (270), (314), (388) ~ (397), (498) ~ (507), (578), (652) ~ (661), (762) ~ (771) in consideration of Definition 5. And I get

\[
(A^{\mu;\nu;\rho})_{\mu;\nu} = \frac{\partial^3 A^\mu}{\partial x^\mu \partial x^\mu \partial x^\nu} \quad (824)
\]

from (190), (234), (278), (322), (408) ~ (417), (518) ~ (527), (586), (672) ~ (681), (782) ~ (791) in consideration of Definition 5. In addition,

\[
A^{\mu;\nu;\rho} = \frac{\partial^3 A^\mu}{\partial x^\nu \partial x^\nu \partial x^\rho} \quad (825)
\]

can rewrite (822), (823), (824) using Definition 2, Definition 3 because the second term doesn’t exist in the right side of (822), (823), (824). Because (825) doesn’t change the form of the equation even if all coordinate systems satisfies Binary Law, I get the conclusion that (825) satisfies Binary Law. In other words, (825) is an equation of the tensor satisfying Binary Law. Because (825) is an equation provided from Definition 8, (825) is the equation that was rewritten so that Definition 8 is an equation of the tensor satisfying Binary Law.

– End Proof –

6. Property of Tensor Satisfying Binary Law about the Third-Order Covariant Derivative of the Contravariant Vector

Proposition 7 If \(x^\mu \neq x^\mu, \overline{x}^\nu \neq x^\nu, \overline{x}^\nu = x^\nu, \overline{x}^\nu = x^\nu\) is established, \(\{x^\mu\}_M = \sin \sqrt{M} x^\nu\) is established.

Proof: I change in \(A^\mu\) to \(x^\mu\) in Proposition 6 and get

\[
x^{\mu;\nu;\rho;\tau} = \frac{\partial^3 x^\mu}{\partial x^\nu \partial x^\nu \partial x^\rho \partial x^\tau} \quad (826)
\]

I express the right side of (826) in

\[
\frac{\partial^3 x^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} = M. \quad (827)
\]

I get
\[ \frac{\partial^3 x^1}{\partial x^1 \partial x^1 \partial x^1} = M, \quad \frac{\partial^3 x^1}{\partial x^2 \partial x^2 \partial x^2} = M, \quad \frac{\partial^3 x^2}{\partial x^1 \partial x^1 \partial x^1} = M, \quad \frac{\partial^3 x^2}{\partial x^2 \partial x^2 \partial x^2} = M \]  
(828)

From (827) if a dimensional number is 2. I get

\[ \int \frac{\partial^3 x^1}{\partial x^1 \partial x^1 \partial x^1} \cdot d^3 x \int M d^2 x \cdot \int \frac{\partial^3 x^1}{\partial x^2 \partial x^2 \partial x^2} \cdot d^2 x = \int M d^2 x, \]

\[ \int \frac{\partial^3 x^2}{\partial x^1 \partial x^1 \partial x^1} \cdot d^3 x \int M d^2 x \cdot \int \frac{\partial^3 x^2}{\partial x^2 \partial x^2 \partial x^2} \cdot d^2 x = \int M d^2 x \]  
(829)

From (828). And I get

\[ \frac{\partial^2 x^1}{\partial x^1 \partial x^1} = M x^1, \quad \frac{\partial^2 x^1}{\partial x^2 \partial x^2} = M x^2, \quad \frac{\partial^2 x^2}{\partial x^1 \partial x^1} = M x^1, \quad \frac{\partial^2 x^2}{\partial x^2 \partial x^2} = M x^2 \]  
(830)

From (829). I get

\[ \frac{\partial^2 x^\mu}{\partial x^\nu \partial x^\nu} = M x^\nu \]  
(831)

From (830). I get

\[ \frac{\partial^2 x^\mu}{\partial x^\nu \partial x^\nu} = -M x^\mu \]  
(832)

From (831), Definition 3. And I get

\[ \frac{\partial^2 x^1}{\partial x^1 \partial x^1} = -M x^1, \quad \frac{\partial^2 x^1}{\partial x^2 \partial x^2} = -M x^1, \quad \frac{\partial^2 x^2}{\partial x^1 \partial x^1} = -M x^2, \quad \frac{\partial^2 x^2}{\partial x^2 \partial x^2} = -M x^2 \]  
(833)

From (832). I get

\[ x^1 = \sin \sqrt{M} x^1, \quad x^1 = \sin \sqrt{M} x^2, \quad x^2 = \sin \sqrt{M} x^1, \quad x^2 = \sin \sqrt{M} x^2 \]  
(834)

From (833). And I get
\[ \{x^\mu\}_M = \sin \sqrt{M} x^\nu \quad (835) \]

from (834), Definition 12.

\[ \text{– End Proof –} \]

7. Discussion

About Proposition 7

If \( M > 0 \) is established, (833) doesn’t change. On the other hand, (833) becomes

\[ \frac{\partial^2 x^1}{\partial x^1 \partial x^1} = 0, \quad \frac{\partial^2 x^1}{\partial x^2 \partial x^2} = 0, \quad \frac{\partial^2 x^2}{\partial x^1 \partial x^1} = 0, \quad \frac{\partial^2 x^2}{\partial x^2 \partial x^2} = 0 \quad \text{if } M=0 \text{ is established. Furthermore, (833) becomes} \]

\[ \frac{\partial^2 x^1}{\partial x^1 \partial x^1} = Mx^1, \quad \frac{\partial^2 x^1}{\partial x^2 \partial x^2} = Mx^1, \quad \frac{\partial^2 x^2}{\partial x^1 \partial x^1} = Mx^2, \quad \frac{\partial^2 x^2}{\partial x^2 \partial x^2} = Mx^2 \quad \text{if } M < 0 \text{ is established. Thus, (833) is} \]

Established when \( M > 0 \) is established. In other words, \( \{x^\mu\}_M \) can have a property of waving when \( M > 0 \) is established.

References


